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Rapport de stage

Master 2 Recherche Mathématiques Appliquées

# The adaptive fused lasso regression and its application on microarrays CGH data

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# 1 Introduction

Statistics has been constantly challenged by issues raised in biological sciences. In the early days many problems came from agricultural sciences and genetics, and the explosion of high-throughput technologies in *"omics"* molecular biology has extended the scope of statistical needs in the last twenty years. A common characteristics of many *omics* data is their large dimension compared to the relatively small number of samples on which they are measured, leading to the famous "small $n$ large $p$" problem. Consequently statistical fields such as multiple testing, clustering, classification, regression, and dimension reduction have been (re)investigated following questions raised by molecular biologists (see for instance [7]). Sparse methods such as the Lasso for regression or the Cox model ([10, 11, 2]), which estimate a function and select features at the same time, have gained a lot of popularity in high-dimensional statistics in the last 10 years, although their theoretical analysis is much more recent [14, 1]. Their wide diffusion has also been supported by works on optimization. For instance the fused Lasso criterion leads to a convex optimization problem which has been solved to provide a solution in much less time that a standard convex optimizer [4]. Sparse methods have also been extended to the analysis of piecewise-constant 1D profiles with the fused Lasso [12, 4], triggered by applications in CGH profile analysis [13][8].

aCGH arrays have now become popular tools to identify DNA CNV along the genome. These profiles are used to identify genomic markers to improve cancer prognosis and diagnosis. Like gene expression profiles, CNV profiles are characterized by a large number of variables usually measured on a limited number of samples. In this part, we want to develop methods for prognosis or diagnosis of patient outcome from CNV data. Due to their spatial organization along the genome CNV profiles have a particular structure of correlation between variables. This suggests that classical classification methods should be adapted account for this spatial organization. To do so one can use penalties such as the fused Lasso penalty. The Lasso penalty is a regularization technique for simultaneous estimation and variable selection ([10]). It consists in the introduction of a $l_1$-type penalty, which enforces the shrinkage of coefficients. The fused Lasso was introduced by [12] ; it combines a Lasso penalty over coefficients and a Lasso penalty over their difference, thus enforcing similarity between successive features. One drawback of the Lasso penalty is the fact that, since it uses the same tuning parameters for all regression coefficients, the resulting estimators may suffer from an appreciable bias (see [3]). Moreover recent results show that the underlying model must satisfy a nontrivial condition for the Lasso estimator to be consistent (see [15]). Consequently in some cases the Lasso estimator can be non-consistent. To fix this issue, a possibility is to introduce adaptive weights to penalize different coefficients in the $l_1$ penalty, as done in the adaptive Lasso which enjoys oracle properties. Moreover, the adaptive Lasso can be solved by the same efficient algorithm than the one used to solve the Lasso. Our claim is that the fused Lasso estimator will suffer from the same defaults as the Lasso because it is based on the same $l_1$ penalty, suggesting that it may be interesting to adapt the adaptive approach of [15] to the fused Lasso penalty. We will study the possibility to introduce adaptive weights in the fused Lasso penalty, resulting in the adaptive fused Lasso penalty. In this report, we show in details what the adaptive fused Lasso is, we give our attention to a special case of the adaptive fused Lasso, the adaptive fused Lasso signal approximator (A-FLSA), we adapt a path algorithm to solve the A-FLSA and we apply our algorithm on simulated data in order to compare the fused Lasso and the adaptive fused Lasso.

# 2 The fused Lasso regression

## 2.1 Fused Lasso penalty

Consider the standard linear regression model

$$y_i = \alpha^* + \mathbf{x}_i^T \beta^* + \epsilon_i, \quad i = 1, \ldots, n. \tag{2.1}$$

where $\mathbf{x}_j = (x_{1j}, \ldots, x_{nj})^T$ for $j = 1, \ldots, p$ are the regressors, $\alpha^*$ is the constant parameter, $\beta^* = (\beta_1^*, \ldots, \beta_p^*)^T$ are the associated regression coefficients and $y_i$ is the response for the $i$th observation. Let $\mathbf{X} = [\mathbf{x_1}, \ldots, \mathbf{x_p}]$ be the predictor matrix, we also call it *design* matrix, and let $y = (y_1, \ldots, y_n)$ the response vector. We suppose that the errors $\epsilon_i$ are independent and identically-distributed (iid) Guassian random errors with mean 0 and constant variance $\sigma^2$. We also assume that the predictors are standardized to have mean zero and unit variance, and the outcome $y_i$ has mean zero. Hence we don't need an intercept $\alpha^*$ in the model 2.1.
The unknown parameters in the model 2.1 are usually estimated by minimizing the Ordinary Least Squares (OLS) criterion, in which we seek to minimize the residual squared error, but there are two reasons why the data analyst is often not satisfied with the OLS estimates.

1. *prediction accuracy* the OLS estimates often have low bias but large variance.
2. *interpretation* with a large number of predictors, we often would like to determine a smaller subset that exhibits the strongest effects, but it is computationally infeasible to do subset selection in this case.

*Prediction* accuracy can sometimes be improved by shrinking or setting to 0 some coefficients. By doing so we sacrifice a little bias to reduce the variance of the predicted values and hence may improve the overall prediction accuracy. For this, Tibshirani [10] proposed a new technique, called the Lasso, for "least absolute shrinkage and selection operator". It shrinks some coefficients and sets others to 0.

Letting $\hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_p)^T$ the estimated vector of $\beta$, the Lasso estimate $\hat{\beta}$ minimizes the following problem in the “Lagrange” form

$$f(\beta) = \frac{1}{2}\sum_{i=1}^{n}\left(y_i - \sum_{j=1}^{p} x_{ij}\beta_j\right)^2 + \lambda_1 \sum_{j=1}^{p} |\beta_j|, \tag{2.2}$$

where $\lambda_1$ is a nonnegative regularization parameter. The second term in 2.2 is the so-called “$l_1$ penaly”. The Lasso continuously shrinks the coefficients toward 0 as $\lambda_1$ increases, and some coefficients are shrunk to exact 0 if $\lambda_1$ is sufficiently large. The Lasso has a unique solution assuming no two predictors are perfectly collinear. One drawback of the Lasso is the fact that it ignores ordering of the features, of the type of data we are working on in this report, the microarrays comparative genomic hybridization (microarrays CGH). For this purpose, Tibshirani [12] proposed the *fused* Lasso, which combines a Lasso penalty over coefficients and a Lasso penalty of their difference. The fused Lasso criterion is defined by

$$f(\beta) = \frac{1}{2}\sum_{i=1}^{n}\left(y_i - \sum_{j=1}^{p} x_{ij}\beta_j\right)^2 + \lambda_1 \sum_{j=1}^{p} |\beta_j| + \lambda_2 \sum_{j=1}^{p-1} |\beta_j - \beta_{j+1}|. \tag{2.3}$$

The first penalty encourages sparsity in the coefficients, the second penalty encourages sparsity in their differences, therefore as $\lambda_1$ and $\lambda_2$ increase, some coefficients are shrunk to 0 and some consecutive coefficients are shrunk to be equal to each other, thus enforcing similarity between successive features.

## 2.2 Adaptive fused Lasso penalty

Recent studies showed that the resulting estimators of the Lasso may suffer from an appreciable bias. Moreover, they showed that the underlying model must satistfy a nontrivial condition for the Lasso estimator to be consistent (Zou [15]). In his paper [15], Zou proposed a new version of the Lasso, called the adaptive Lasso, where adaptive weights are used for penalizing different coefficients in the $l_1$ penalty. He showed that the adaptive Lasso enjoys the oracle properties.

### 2.2.1 Adaptive Lasso and Oracle properties

Consider the standard linear regression model 2.1. We assume that the data are centered, so the intercept $\alpha^*$ is not included in the regression function. Let $\mathcal{A} = \{1 \leq j \leq p,\ \beta_j^* \neq 0\}$ and $p_0 = |\mathcal{A}|$ the cardinal of $\mathcal{A}$. We assume that $p_0 < p$. Thus the true model depends only on a subset of the predictors. Denote by $\hat{\beta}(\delta)$ the coefficient estimator produced by a fitting procedure $\delta$. Using the language of Fan and Li [3], we call $\delta$ an *oracle* procedure if $\hat{\beta}(\delta)$ (asymptotically) satisfies the following oracle property :

– Consistency in variable selection : $\lim_n \mathbb{P}(\mathcal{A}^* = \mathcal{A}) = 1$,

where $\mathcal{A}^* = \{j : \hat{\beta}_j(\delta) \neq 0\}$.

It has been argued (Fan and Li [3]) that a good procedure should have these oracle properties. Fan and Li [3] studied a class of penalization methods including the Lasso. They showed that the Lasso shrinkage produces biased estimates for the large coefficients, and thus it could be suboptimal in terms of estimation risk. Fan and li [3] conjectured that the oracle propeties do not hold for the Lasso. Then Zou [15] showed that the Lasso variable selection can be inconsistent in some scenarios.

Basing on the aboved, Zou [15] derived a necessary condition for the Lasso variable selection to be consistent. He proposed a new version of the Lasso, called the *adaptive* Lasso where adaptive weights are used for penalizing different coefficients in the $l_1$ penalty. He also showed that the adaptive Lasso enjoys the oracle properties. Let us consider the weighted Lasso, by adding those weights the equation 2.2 that we seek to minimize in the Lasso problem become

$$f(\beta) = \frac{1}{2}\sum_{i=1}^{n}\left(y_i - \sum_{j=1}^{p} x_{ij}\beta_j\right)^2 + \lambda_1 \sum_{j=1}^{p} w_j|\beta_j|, \tag{2.4}$$

where $\mathbf{w} = (w_1, \dots, w_p)$ is a known weights vector. Zou [15] showed that if the weights are data-dependent and cleverly chosen, the weighted Lasso can have the oracle properties. He called this methodology the adaptive Lasso.

So, in order to choose the appropriate weights, consider $\hat{\beta}_{ols}$ the ordinary least squares estimator of $\beta^*$, which is a root-$n$-consistent estimator to $\beta^*$. Pick a $\gamma > 0$, therefore we define the weight vector as $\mathbf{w} = |\hat{\beta}_{ols}|^{-\gamma}$, which is well data-dependent.

### 2.2.2 Adaptive fused Lasso

Now back to the fused Lasso 2.3 proposed by Tibshirani [12], the fused Lasso will suffer from the same defaults as the Lasso since it is based on the same $l_1$ penalty. Therefore, we will introduce some weights in the fused Lasso model, as same as Zou [15] did with the Lasso problem. Precisely we propose the following criterion

$$f(\beta) = \frac{1}{2}\sum_{i=1}^{n}\left(y_i - \sum_{j=1}^{p} x_{ij}\beta_j\right)^2 + \lambda_1 \sum_{j=1}^{p} w_j^{(1)}|\beta_j| + \lambda_2 \sum_{j=1}^{p-1} w_j^{(2)}|\beta_j - \beta_{j+1}|, \tag{2.5}$$

where $\mathbf{w}^{(1)} = |\hat{\beta}_j|^{-\gamma}$, $j = 1, \dots, p$ and $\mathbf{w}^{(2)} = |\hat{\beta}_j - \hat{\beta}_{j+1}|^{-\gamma}$, $j = 1, \dots, p-1$ are the weights vectors. Note that $\hat{\beta}$ here is always the estimate of $\beta^*$ obtained by OLS, $\hat{\beta} = \hat{\beta}_{ols}$. In this report we will pick $\gamma = 1$.

## 2.3 The FLSA and microarrays CGH

The fused Lasso method is especially suitable for coefficients that are constant for an interval and change in jumps, an example is shown in Fig 2.1 below, since the penalty on the absolute values $|\beta_i|$ encourage sparseness and the penalty on $|\beta_i - \beta_{i+1}|$ tends to set neighboring penalties exactly equal to each other.

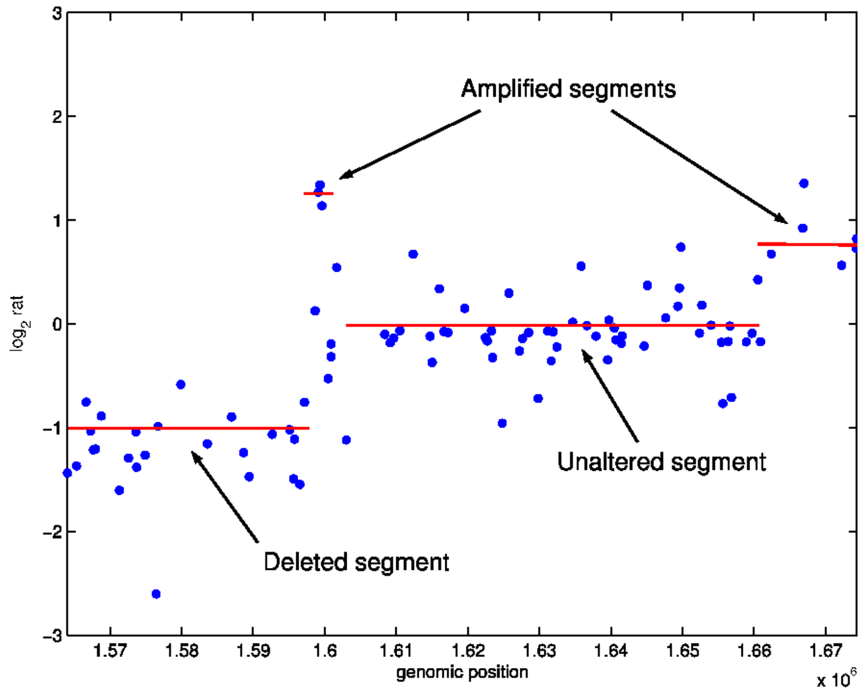


FIG. 2.1 – example of microarray CGH data

We will apply the methods we are studying in this report on comparative genomic hybridization (CGH) data. CGH is a method that identifies DNA copy number gains and losses on chromosomes by making two color fluorescence in situ hybridization at various points of the chromosomes. In this technique, normal and tumor DNA are labeled with fluorescent dyes (e.g. red and green) and using a microarray analysis, regions of increased or decreased fluorescence of one color compared to the other can be identified, indicating gains or losses of DNA at this place of the chromosome. As usual with this type of data, it is very noisy. Therefore, we seek to exploit that gains or losses typically appear for whole regions in the genome and that these changes usually occur in jumps. We can do this by penalizing differences of neighboring coefficients and therefore decrease the noise in the data and improve estimation. We concentrate on the most widely used case for the fused Lasso method, the Fused Lasso Signal Approximator (FLSA). In the FLSA, we assume that we have $\mathbf{X} = \mathbf{I}$ as the predictor matrix and that we have $n = p$. Therefore the loss function 2.3 becomes

$$f(\beta) = \frac{1}{2}\sum_{i=1}^{n}(y_i - \beta_i)^2 + \lambda_1 \sum_{i=1}^{n} |\beta_i| + \lambda_2 \sum_{i=1}^{n-1} |\beta_i - \beta_{i+1}|. \tag{2.6}$$

Every coefficient $\beta_i$ is an estimate of the measurement $y_i$ taken at position $i$ (which we assume to be ordered along the chromosome). Apart from the Lasso penalty $\lambda_1 \sum_{i=1}^{n} |\beta_i|$, the additional penalty placed on the difference between neighboring coefficients is $\lambda_2 \sum_{i=1}^{n-1} |\beta_i - \beta_{i+1}|$. An example of CGH measurements in lung cancer can be seen in Fig 2.2. The red line are the estimates for penalty parameters $\lambda_1 = 0$ and $\lambda_2 = 2$. We can see that starting around measurement 150, the CGH results are on average below 0, indicating a loss of DNA in this region.

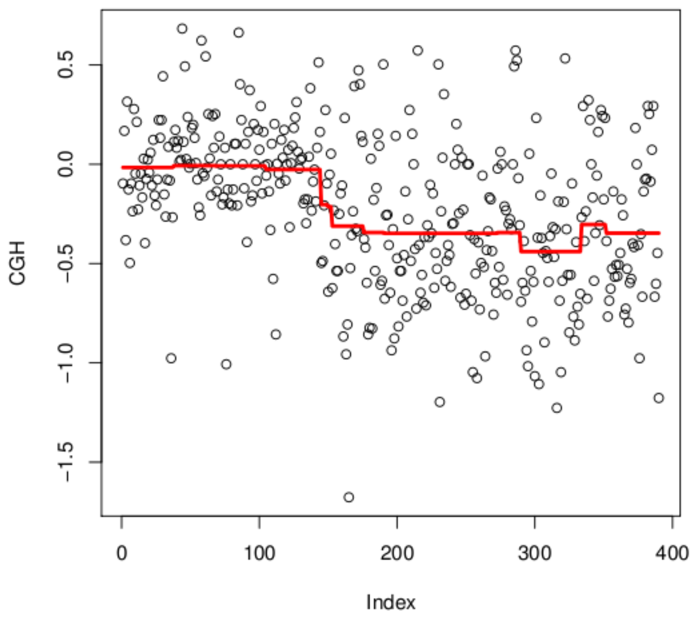


FIG. 2.2 – Example using the one-dimensional Fused Lasso Signal Approximator on lung cancer CGH data.

## 2.4 The A-FLSA

As we already mentioned in section 2.2.3 about the fused Lasso model, it will suffer from some defaults, we proposed to add weights in the criterion in order to avoid these defaults, so we will do the same for the FLSA, we will add those weights in the equation 2.6. We will call it the "Adaptive Fused Lasso Signal Approximator" (A-FLSA). The loss function 2.6 we seek to minimize becomes

$$f(\beta) = \frac{1}{2}\sum_{i=1}^{n}(y_i - \beta_i)^2 + \lambda_1 \sum_{i=1}^{n} w_i^{(1)}|\beta_i| + \lambda_2 \sum_{i=1}^{n-1} w_i^{(2)}|\beta_i - \beta_{i+1}|, \tag{2.7}$$

where the weights are the same defined above.
In the next chapter, we will explore a path algorithm to solve the A-FLSA, then we will use cross-validation to find the optimal values of $\lambda_1$ and $\lambda_2$.

### 2.4.1 The A-FLSA's oracle properties

To verify that we can use the A-FLSA, we must be sure that it enjoys the oracle properties (see section 2.2.1). Zou [15] gave a necessary condition to the consistency of the Lasso selection, it also can be a necessary condition of the A-FLSA since it is based on the same penalty. We assume that we have the model presented in 2.1 without the intercept, and that $\frac{1}{n}\mathbf{X}^T\mathbf{X} \to \mathbf{C}$ where $\mathbf{C}$ is a positive definite matrix.
Without loss of generality, assume that $\mathcal{A} = \{1, 2, \ldots, p_0\}$. Recall that $\mathcal{A} = \{j : \beta_j^* \neq 0\}$. Let

$$\mathbf{C} = \begin{bmatrix} \mathbf{C_{11}} & \mathbf{C_{12}} \\ \mathbf{C_{21}} & \mathbf{C_{22}} \end{bmatrix}$$

where $\mathbf{C_{11}}$ is a $p_0 \times p_0$ matrix. Let $\hat{\beta}$ the A-FLSA estimates, obtained by minimizing equation 2.7. Recall that $\mathcal{A}^* = \{j : \hat{\beta}_j \neq 0\}$. The necessary condition presented in [15] is the following

- Suppose that $\lim_n \mathbb{P}(\mathcal{A}^* = \mathcal{A}) = 1$. Then there exists some sign vector $\mathbf{s} = (s_1, \ldots, s_{p_0})^T$, $s_j = 1$ or $-1$, such that

$$|\mathbf{C_{21}}\mathbf{C_{11}^{-1}}\mathbf{s}| \leq 1 \tag{2.8}$$

Zou [15] showed that if condition 2.8 fails, then the Lasso variable selection is inconsistent. He found an example where condition 2.8 cannot be satistied.

As for the A-FLSA, we have always $\mathbf{X} = \mathbf{I}$ the design matrix. In this case, the condition 2.8 is satisfied since $|\mathbf{C_{21}}\mathbf{C_{11}^{-1}}\mathbf{s}|$ in this case is always 0, thus it is $\leq 1$. Therefore, we satisfy the necessary condition but we can say nothing more about the consistency of the A-FLSA, so we are not quite sure that the A-FLSA enjoys the oracle properties.
In perspective, it will be interesting to find a necessary condition for the consistency of the adaptive fused Lasso variable selection in the case of random $\mathbf{X}$ as a design matrix,

especially trying to find what kind of hypotheses we need to propose, since we penalize an additional term in the fused Lasso, the absolute value of difference of coefficients.

# 3 Path algorithm to solve the A-FLSA

We consider the generic regularized optimization problem $\hat{\beta}(\lambda_1) = argmin_{\beta} L(y, X\beta) + \lambda_1 J(\beta)$ that is the case of the Lasso problem. Efron et al.[2] have shown that for the Lasso, that is if $L$ is squared error loss and $J(\beta) = \|\beta\|_1 = \sum_{i=1}^{p} |\beta_i|$ is the $l_1$ norm of $\beta$ if its length is $p$, the optimal coefficient path is piecwise linear, i.e., $\partial\hat{\beta}(\lambda_1)/\partial\lambda_1$ is piecewise constant. Rosset and Zhu [9] derived a general characterization of the properties of (loss $L$, penalty $J$) pairs which give piecewise linear coefficient paths. Such pairs allow for efficient generation of the full regularized coefficient paths.

## 3.1 Piecewise linear regularized solution path

Regularization is an essential component in modern data analysis, in particular when the number of predictors is large, possibly larger than the number of observations, and non-regularized fitting is likely to give badly over-fitted and useless models. We will concentrate on a special case of the generic regularized optimization problems which is the Lasso, and later on its generalizations, the fused Lasso and the adaptive fused Lasso. The inputs we have in the Lasso are :

- A data sample $\mathbf{X}$ and the response vector $y$ as described in chapter 2.
- A convex non-negative loss functional $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.
- A convex non-negative penalty functional $J : \mathbb{R}^p \to \mathbb{R}$, with $J(0) = 0$.

We want to find :

$$\hat{\beta}(\lambda_1) = argmin_{\beta \in \mathbb{R}^p} L(y, X\beta) + \lambda_1 J(\beta), \tag{3.1}$$

where $\lambda_1 \geq 0$ is the regularization parameter, $\lambda_1 = 0$ corresponds to no regularization, while $\lim_{\lambda_1 \to \infty} \hat{\beta}(\lambda_1) = 0$.

The Lasso uses squared error loss as $L$ and the $l_1$ norm as the penalty $J$, therefore we write 3.1 as,

$$\hat{\beta}(\lambda_1) = \min_{\beta} \sum_{i=1}^{n} (y_i - \mathbf{x}_i^T \beta)^2 + \lambda_1 \sum_{i=1}^{p} |\beta_i|. \tag{3.2}$$

Many "modern" methods for machine learning, signal processing and statistical modeling can also be cast in the framework of regularized optimization. For example, the regularized support vector machine, the ridge regression and the penalized logistic regression.

For the coefficient paths to be piecewise linear, we require that $\frac{\partial \hat{\beta}(\lambda_1)}{\partial \lambda_1} / \| \frac{\partial \hat{\beta}(\lambda_1)}{\partial \lambda_1} \|$ is a piecewise constant vector as a function of $\lambda_1$. [9] showed two sufficient conditions for piecewise linearity :

- $L$ is piecewise quadratic as a funtion of $\beta$ along the optimal path $\hat{\beta}(\lambda_1)$, when $\mathbf{X}, y$ are assumed constant at their sample values.
- $J$ is piecewise linear as a function of $\beta$ along this path.

We concentrate our attention on (loss $L$, penalty $J$) pairings where the optimal path $\hat{\beta}(\lambda_1)$ is *piecewise linear* as a funtion of $\lambda_1$, i.e., $\exists \lambda_1^{(0)} = 0 < \lambda_1^{(1)} < \ldots < \lambda_1^{(m)} = \infty$ and $\gamma_0, \gamma_1, \ldots, \gamma_{m-1} \in \mathbb{R}^p$ such that $\hat{\beta}(\lambda_1) = \hat{\beta}(\lambda_1^{(k)}) + (\lambda_1 - \lambda_1^{(k)})\gamma_k$ for $\lambda_1^{(k)} \leq \lambda_1 \leq \lambda_1^{(k+1)}$. Such models are attractive because they allow us to generate the whole regularized path $\hat{\beta}(\lambda_1)$, for $0 \leq \lambda_1 \leq \infty$, simply by sequentially calculation the "step sizes" between each two consecutive $\lambda_1$ values and the "directions" $\gamma_1, \ldots, \gamma_{m-1}$. A canonical example is the Lasso 3.2. In 2004, [2] have shown that the piecewise linear coefficient paths property holds for the Lasso.

As for the fused Lasso, it is obvious that the piecewise linear coefficient paths property holds. The difference we have is that $\hat{\beta}$ depends on two values $\lambda_1$ and $\lambda_2$ instead of just $\lambda_1$, with adding a second term on the regularized optimization problem, which is the penalty on the differences of consecutive values of $\beta$. Let us call $J_1(\beta) = \sum_{i=1}^{p} |\beta_i|$ and $J_2(\beta) = \sum_{i=1}^{p-1} |\beta_i - \beta_{i+1}|$, in the fused Lasso we want to find

$$\hat{\beta}(\lambda_1, \lambda_2) = argmin_{\beta \in \mathbb{R}^p} L(y, X\beta) + \lambda_1 J_1(\beta) + \lambda_2 J_2(\beta). \tag{3.3}$$

As for the adaptive fused Lasso, nothing will be changed since the weights we consider are constant and do not affect on the piecewise linearity of $J_1(\beta)$ and $J_2(\beta)$. Therefore, we will use a path algorithm to solve the A-FLSA.

## 3.2 Path algorithm for the A-FLSA

In his paper, Hoefling [6] described a path algorithm that solves the FLSA. We adapted this algorithm in order to solve the A-FLSA. We will desribe the algorithm in this section. An important clue of this algorithm is that due to the simple structure of the loss function 2.7, it is possible in this case to obtain the solution for any value of ($\lambda_1$,$\lambda_2$) by simple soft-thresholding of the solution obtained for (0, $\lambda_2$). To be more precise, the following theorem holds :

**Theorem 1.** *Assume we have $X = I$ and that the solution for $\lambda_1 = 0$ and $\lambda_2 > 0$ is known and denote it by $\hat{\beta}(0, \lambda_2)$. Then the solution for $\lambda_1 > 0$ is*

$$\begin{aligned}\hat{\beta}_i(\lambda_1, \lambda_2) &= S(\hat{\beta}_i(0, \lambda_2), w_1^{(i)}\lambda_1) \\ &= sign(\hat{\beta}_i(0, \lambda_2))\ (|\hat{\beta}_i(0, \lambda_2)| - w_1^{(i)}\lambda_1)^+ \quad for\ \ i = 1, \cdots, p\end{aligned}$$

*where $y^+ = max(0, y)$.*

The proof of this theorem is presented in the appendix. For the rest of this section, we assume that $\lambda_1 = 0$. The algorithm presented is a path algorithm that finds the solution for all possible values of $\lambda_2$. Any solution for a $\lambda_1 \neq 0$ can then be obtained by simply soft-thresholding as shown above. Therefore the criterion we want to minimize becomes

$$L(y, \beta) = \frac{1}{2}\sum_{i=1}^{n}(y_i - \beta_i)^2 + \lambda_2 \sum_{i=1}^{n-1} w_i^{(2)}|\beta_i - \beta_{i+1}|, \tag{3.4}$$

where $w_i^{(2)} = |\beta_i - \beta_{i+1}|^{-1}$.
The path algorithm will start by setting $\lambda_2 = 0$ and then increase it until all coefficients $\beta_i$ have the same value. For increasing $\lambda_2$ neighboring coefficients are forced to be equal to each other. Once this happens, these coefficients are being *fused* and subsequently treated as a single variable for increasing $\lambda_2$. In order to be able to do this, a special result holds for the FLSA, that makes the algorithm especially simple and also has been presented and proved in [4]. It states that if coefficients are fused at $\lambda_2^0$ , then these coefficients will also be fused for any $\lambda_2 > \lambda_2^0$. To be more precise

**Theorem 2.** *Let $\beta_k(0, \lambda_2)$ be the optimal solution to the A-FLSA problem for coefficient k and penalty parameter $\lambda_2$. Then if for some k and $\lambda_2^0$ it holds that $\beta_k(0, \lambda_2^0) = \beta_{k+1}(0, \lambda_2^0)$, then for any $\lambda_2 > \lambda_2^0$ it holds that $\beta_k(0, \lambda_2) = \beta_{k+1}(0, \lambda_2)$.*

## The algorithm

In order to develop the algorithm, we first have to define what exactly the sets of fused coefficients are.

**Definition 1.** *([6]) Let $F_i$ $i = 1, \cdots, n_F(\lambda_2)$ be the set of coefficients at $\lambda_2$ that are considered to be fused, where $n_F(\lambda_2)$ is the number of such sets. In order for these sets to be valid, we will write every set $F_i$ in the form of $F_i = \{k | l_i \leq k \leq u_i\}$ and the following statements have to hold :*

- $\cup_{i=1}^{n_F(\lambda_2)} F_i = 1, \cdots, n.$
- $F_i \cap F_j = \emptyset, \ \ i \neq j.$
- *Assuming the $F_i$ are ordered, for every $k, l \in F_i$ we have $\beta_k(\lambda_2) = \beta_l(\lambda_2)$ and for $k \in F_i, l \in F_{i+1}$ it holds that $\beta_k(\lambda_2) \neq \beta_l(\lambda_2)$.*

For notational convenience, we will write $\beta_{F_i}(\lambda_2)$ for any $\beta_k(\lambda_2)$ with $k \in F_i$ and also suppress the dependency of $F_i$ onto $\lambda_2$. Using this definition of fused sets, let us now turn to the algorithm.
The algorithm is too simple due to Theorem 1 and Theorem 2. First, we need a starting point for the path algorithm and as the optimal solution is known for $\lambda_2 = 0$, which is just $\beta_k(0) = y_k$ for all $k$, we use it to begin the path. Then the algorithm calculates step by step when two neighboring sets have equal coefficients and merges them.

Now let us introduce the loss function that incorporates the fused sets $F_i$, we will call it $L_{F,\lambda_2}$. This is done by taking the loss function $L$ in equation 3.4 and replacing $\beta_k$ by $\beta_{F_i}$ for $k \in F_i$. This constrained loss function is then

$$L_{F,\lambda_2}(y, \beta) = \frac{1}{2} \sum_{i=1}^{n_F(\lambda_2)} \left( \sum_{j \in F_i} (y_j - \beta_{F_i})^2 \right) + \lambda_2 \sum_{i=1}^{n_F(\lambda_2)-1} \frac{|\beta_{F_i} - \beta_{F_{i+1}}|}{|\hat{\beta}_{maxF_i} - \hat{\beta}_{minF_{i+1}}|}. \tag{3.5}$$

Note that $F_i$ contains the indices of the coefficients $\beta_k$ that are fused at the corresponding value $\lambda_2$, thus the term $|\hat{\beta}_{maxF_i} - \hat{\beta}_{minF_{i+1}}|^{-1}$ is the appropriate weight to adapt $|\beta_{F_i} - \beta_{F_{i+1}}|$, so $\hat{\beta}_{maxF_i}$ is the OLS estimate of $\beta^*_{maxF_i}$.

This constrained loss function 3.5 is always differentiable with respect to $\beta_{F_i}$ due to the assumptions on the sets F. To get an optimal solution of $\beta_{F_i}$, we will calculate the derivative of $L_{F,\lambda_2}$ with respect to $\beta_{F_i}$ and find its root. The derivative of $L_{F,\lambda_2}$ for $i = 1, \ldots, n_F(\lambda_2)$ is

$$\frac{\partial L_{F,\lambda_2}}{\partial \beta_{F_i}} = (\#F_i)\beta_{F_i} - \sum_{j \in F_i} y_j + \lambda_2 \frac{sign(\beta_{F_i} - \beta_{F_{i-1}})}{|\hat{\beta}_{maxF_{i-1}} - \hat{\beta}_{minF_i}|} + \lambda_2 \frac{sign(\beta_{F_i} - \beta_{F_{i+1}})}{|\hat{\beta}_{maxF_i} - \hat{\beta}_{minF_{i+1}}|}, \tag{3.6}$$

where $\#F_i$ is the cardinal of $F_i$. We set $sign(\beta_{F_1} - \beta_{F_0}) = 0$ and $sign(\beta_{F_{n_F(\lambda_2)}} - \beta_{F_{n_F(\lambda_2)+1}}) = 0$ since $\beta_{F_0}$ and $\beta_{F_{n_F(\lambda_2)+1}}$ do not exist.
Then by putting $\partial L_{F,\lambda_2}/\partial\beta_{F_i} = 0$ the optimal value of $\beta_{F_i}$ for $i = 1, \ldots, n_F(\lambda_2)$ is

$$\beta_{F_i}(\lambda_2) = \frac{1}{\#F_i}\left[\sum_{j\in F_i} y_j - \lambda_2(\frac{sign(\beta_{F_i} - \beta_{F_{i-1}})}{|\hat{\beta}_{maxF_{i-1}} - \hat{\beta}_{minF_i}|} + \frac{sign(\beta_{F_i} - \beta_{F_{i+1}})}{|\hat{\beta}_{maxF_i} - \hat{\beta}_{minF_{i+1}}|})\right]. \tag{3.7}$$

We will calculate the derivative of $\beta_k(\lambda_2)$ with respect to $\lambda_2$ and we will see that these are actually constant, so that the resulting solution path is a piecewise linear function [9] where the breakpoints occur when two sets of coefficients are being fused. The derivative of $\beta_k(\lambda_2)$ for $i = 1, \ldots, n_F(\lambda_2)$ is

$$\frac{\partial\beta_{F_i}}{\partial\lambda_2} = -\frac{1}{\#F_i}\left(\frac{sign(\beta_{F_i} - \beta_{F_{i-1}})}{|\hat{\beta}_{maxF_{i-1}} - \hat{\beta}_{minF_i}|} + \frac{sign(\beta_{F_i} - \beta_{F_{i+1}})}{|\hat{\beta}_{maxF_i} - \hat{\beta}_{minF_{i+1}}|}\right). \tag{3.8}$$

These derivatives are constant as long as the sets of fused coefficients do not change. By Theorem 2, we know that only way to change the sets of fused coefficients is to merge two sets as they can never split for increasing $\lambda_2$. Therefore, the solution path is piecewise linear and for increasing $\lambda_2$ the breakpoint occurs when two groups fuse. Thus, it is easy to calculate the next breakpoint, which occurs when neighboring sets have the same coefficients. In order to do this, define

$$h_{i,i+1}(\lambda_2) = \frac{\beta_{F_i}(\lambda_2) - \beta_{F_{i+1}}(\lambda_2)}{\frac{\partial\beta_{F_{i+1}}}{\partial\lambda_2} - \frac{\partial\beta_{F_i}}{\partial\lambda_2}} + \lambda_2 \text{ for } i = 1, \ldots, n_F(\lambda_2) - 1.$$

which is the value for $\lambda_2$ at which the coefficients of the sets $F_i$ and $F_{i+1}$ have the same value and can be fused, assuming that no other coefficients become fused before that. If $h_{i,i+1}(\lambda_2) < \lambda_2$, these values are being ignored as the two groups $F_i$ and $F_{i+1}$ are actually moving apart for increasing $\lambda_2$. The next value at which coefficients are fused is therefore the hitting time

$$h(\lambda_2) = \min_{h_{i,i+1}(\lambda_2)>\lambda_2} h_{i,i+1}(\lambda_2).$$

As we taking the minimum, it is only defined if there is at least one $h_{i,i+1}(\lambda_2) > \lambda_2$. From equation (2.2) with $\lambda_1 = 0$ we can easily see that for $\lambda_2 \to \infty$ the solution is $\beta_k = \frac{1}{n}\sum_{l=1}^{n} y_l$ for all $k$, thus only one group exists for large $\lambda_2$. Therefore, if $n_F(\lambda_2) \geq 2$, then there exists $h_{i,i+1}(\lambda_2) > \lambda_2$ and therefore $h(\lambda_2)$ is defined. Based on this results, we write the details of the algorithm that provides the entire solution path and the can be found in Algorithm 1.

**Algorithm 1**: One-dimensional FLSA path algorithm

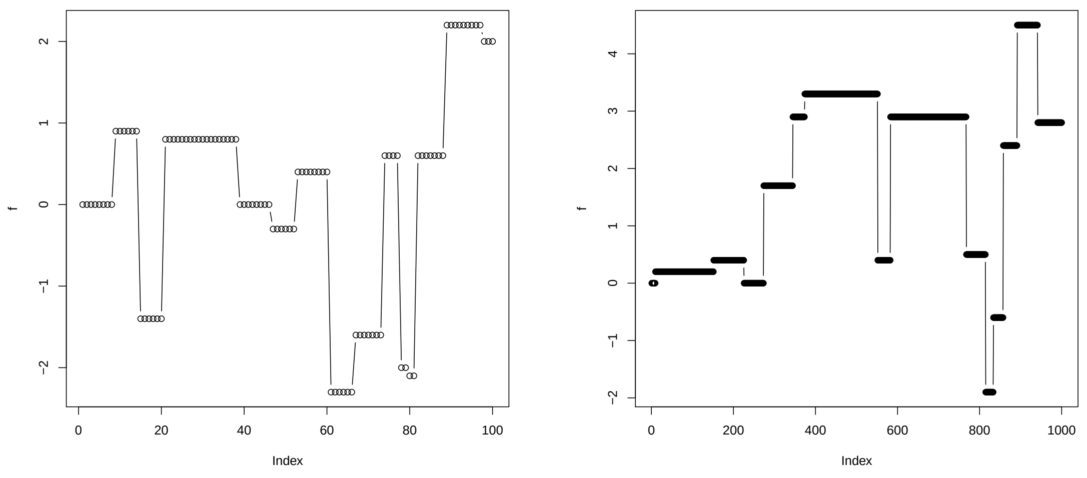
```
begin
    $\lambda_2 = 0$;
    $\beta_k = y_k$ for $k = 1, \cdots, n$;
    $F_i = \{i\}$ for $i = 1, \cdots, n$;
    $n_F = n$;
end
while $n_F > 1$ do
    Calculate next hitting time $h(\lambda_2)$;
    Let $(i_0(\lambda_2), i_0(\lambda_2) + 1) = \text{argmin}_{h_{i,i+1} > \lambda_2} h_{i,i+1}(\lambda_2)$ be the indices of the sets to
    be fuse next;
    Fuse the two sets $F_{i_0(\lambda_2)}$ and $F_{i_0(\lambda_2)+1}$;
    Set : $\lambda_2 = h(\lambda_2)$;
    Update the values for $\beta_k(\lambda_2)$, $\frac{\partial \beta_k(\lambda_2)}{\partial \lambda_2}$ and set $n_F = n_F - 1$;
end
```

# 4 Numerical Results

We will compare the two methods FLSA and A-FLSA by applying both on simulated data, the algorithm of the first method was implemented by [6], we adapted it to apply the second method. In order to find the optimal values of $\lambda_1$ and $\lambda_2$ we used cross validation method. We will define some values that help us to compare the obtained results.

## 4.1 Simulations and noise

We simulated similar data to the real microarrays CGH data, we have already mentioned in section 2.3 that CGH data have the form of segments, like Fig. 2.1 shows. We simulated two datasets, we consider $n = 100$ in the first one and $n = 1000$ in the second. The two simulated datasets are shown in the figure below.



(a) 100 simulated $\beta^*$

(b) 1000 simulated $\beta^*$

FIG. 4.1 – Our two simulated datasets

Back to the initial model (2.1), we supposed that $\mathbf{X} = \mathbf{I}$ as the design matrix and that $p = n$, therefore we write the model as

$$y_i = \beta_i^* + \epsilon_i \qquad i = 1, \ldots, n.$$

We also supposed that the errors $\epsilon_i$ are iid with mean 0 and constant variance $\sigma^2$. So we must add some noise on the simulated datasets. To do so, we choose several level of noise, we calculate three different values of $\sigma$, corresponding to three values of Signal-to-Noise Ratio (SNR) given respectively 3, 5 and 7. The relation between $\sigma$ and SNR is the following

$$\sigma^2 = \frac{1}{n} \frac{\|\beta^*\|_2^2}{SNR^2}.$$

where $\|\beta^*\|_2^2 = \sum_{i=1}^n (\beta_i^*)^2$. Therefore, if SNR=3, $\sigma$ is high and the model is very noisy. If SNR=7, the model is less noisy. For each simulated dataset, we repeated the adding noise process 500 times, for three different levels of noise.

## 4.2 Cross Validation

Now in order to find the optimal values of $\lambda_1$ and $\lambda_2$ in both of FLSA and A-FLSA, we tried two methods, the Bayesian Information Criterion (BIC) and cross validation. After some tries on many randomly choosed simulations, we found that BIC does not give us the optimal values, so we used cross validation.

Cross validation is a technique for assessing how the results of a statistical analysis will generalize to an independent data set. It is mainly used in settings where the goal is prediction, and one wants to estimate how accurately a predictive model will perform in practice. Cross validation involves partitioning a sample of data into complementary subsets, performing the analysis on one subset (called the training set), and validating the analysis on the other subset (called the validation set or testing set).
Let $y_i$ one of the 500 simulations and we want to find the optimal values of $\lambda_1$ and $\lambda_2$ for whose we have the best estimated $\hat{\beta}$ using either FLSA or A-FLSA. For instance, suppose that we are in the case of $n = 100$, so we divide $y_i$ into two subsets. The first subset is the testing set, we will call it $y^{(T)}$, and it is in the form

$$y^{(T)} = \{y_i, i = 2k, k = 1, \ldots, 50\}.$$

The second subset is the training set, we will call it $y^{(L)}$, it contains the other elements of $y_i$, therefore

$$y^{(L)} = \{y_i, i = 2k + 1, k = 0, \ldots, 49\}.$$

We will apply the regression FLSA and A-FLSA on the training set on a grill of values of $\lambda_1$ and $\lambda_2$. For each pair ($\lambda_1$,$\lambda_2$) we find the estimated parameters, let $\hat{\beta}^L_{\lambda_1,\lambda_2}$ the vector of these estimates. Then we calculate the Mean Squared Error (MSE) which is in this case

$$MSE_{(\lambda_1,\lambda_2)} = \sum_{i=1}^{50}(y_i^{(T)} - \hat{\beta}^{(L)}_{i_{\lambda_1,\lambda_2}})^2 \qquad \text{for every pair } (\lambda_1, \lambda_2).$$

Therefore, the optimal values of $\lambda_1$ and $\lambda_2$ correspond to the minimal value of MSE.

$$(\lambda_1, \lambda_2) = \text{argmin}_{(\lambda_1,\lambda_2)} MSE.$$

## 4.3 Comparing the performance of FLSA and A-FLSA

Our objective is to compare the performance of the two methods FLSA and A-FLSA, i.e., it is about multiple tests comparison. Statisticians were always interested by finding an error rate to compare multiple tests, one of the most important rates discovered is the false discovery rate (FDR). We will use this rate and another one, the false negative rate (FNR) to compare the performance of FLSA and A-FLSA on simulated data. We will also compare the important mean squared error (MSE) in both of FLSA and A-FLSA.

### 4.3.1 False discovery rate

In many multiplicity problems the number of erroneous rejections should be taken into account and not only the question whether any error was made. Yet, at the same time, the seriouness of the loss incurred by erroneous rejections is inversely related to the number of hypotheses rejected. From this point of view, a desirable error rate to control may be the expected proportion of errors among the rejected hypotheses, which is the false discovery rate (FDR).
To be more specific, consider the problem of testing simultaneously $m$ (null) hypotheses, of which $m_0$ are true, R is the number of hupotheses rejected. Table 4.1 summarizes the situation in a traditional form.

TAB. 4.1 – Number of errors committed when testing $m$ null hypotheses

| | Declared non-significant | Declared significant | Total |
|---|---|---|---|
| `True null hypotheses` | U | V | $m_0$ |
| `Non-true null hupotheses` | T | S | $m - m_0$ |
| | $m$ - R | R | $m$ |

The specific $m$ hypotheses are assumed to be known in advance. R is an observable random variable ; U, V, S and T are unobservable random variables. The proportion of errors committed by falsely rejecting null hypotheses can be viewed through the random variable Q = V/(V + S), the proportion of the rejected null hypotheses which are erroneously rejected. Q is an unobserved (unknown) random variable. FDR is defined to be the expectation of Q,

$$FDR = \mathbb{E}(Q) = \mathbb{E}[\frac{V}{V+S}] = \mathbb{E}(\frac{V}{R}).$$

Classically, in segmentation we compute some metrics and we calculate FDR using these metrics. To be more clear, we will explain how to evaluate the performance of a classifier, note that we are not in a classification problem, segmentation is what does interest us, but this explanation will helps us to understand where do these metrics come from.

## Classifier performance

A classification model (classifier or diagnosis) is a mapping of instances into a certain class. Let us consider a two-class prediction problem (binary classification), in which the outcomes are labeled either as positive (p) or negative (n) class. There are four possible outcomes from a binary classifier. If the outcome from a prediction is p and the actual value is also p, then it is called a *true positive* (TP) ; however if the actual value is n then it is said to be a *false positive* (FP). Conversely, a *true negative* has occurred when both the prediction outcome and the actual value are n, and *false negative* is when the prediction outcome is n while the actual value is p. To get an appropriate example in a real-world problem, consider a diagnostic test that seeks to determine whether a person has a certain disease. A false positive in this case occurs when the person tests positive, but actually does not have the disease. A false negative, on the other hand, occurs when the person tests negative, suggesting they are healthy, when they actually do have the disease.
Given a classifier and a set of instances (the test set), a two-by-two confusion matrix

(also called contingency table) can be constructed representing the dispositions of the set of instances. This matrix forms the basis for many common metrics.

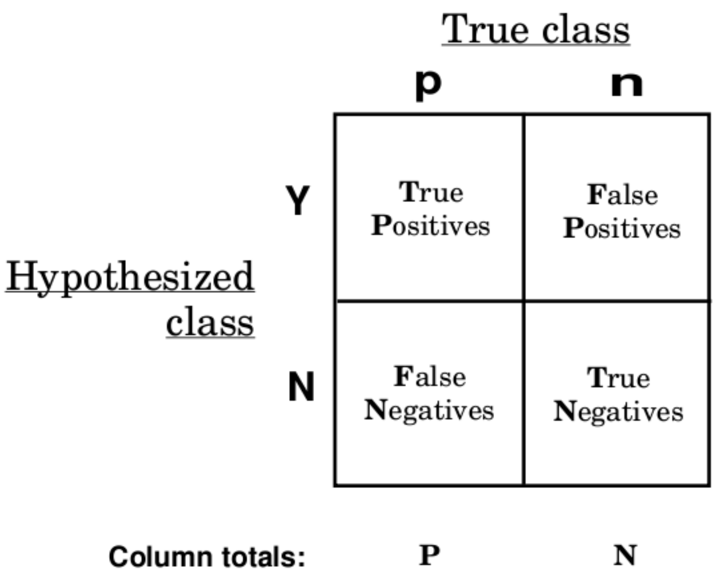


FIG. 4.2 – Confusion matrix

Looking at Table 4.1 and Figure 4.2, we can declare that,
– $m_0$ is the number of true null hypotheses.
– $m - m_0$ is the number of false null hypotheses.
– U is the number of true negatives.
– V is the number of false positives.
– T is the number of false negatives.
– S is the number of true positives.

Now we can define what exactly FDR is in our situation,

$$FDR = \frac{FP}{FP + TP}$$

Another error rate is defined and can be used, but staticians do not give it such big importance, is the false negative rate (FNR), which is

$$FNR = \frac{FN}{FN + TN}$$

### 4.3.2 Performance of FLSA and A-FLSA

Now back to FLSA and A-FLSA, we have two-class prediction problem, but since we putted two penalties, the first one on the absolute values of coefficients, the second one on their difference, therefore for every metric of the described above, we will calculate two values, one due the first penalty and one due the second penalty. Recall that we have the model 2.1, so $\beta^*$ is the actual value and $\hat{\beta}$ is the outcome, therefore

$$\text{False positives}\begin{cases} FP1 = \#\{j:\ \beta^*_j = 0\ \&\ \hat{\beta}_j \neq 0\}\ \text{(The cardinal of this set)} \\ FP2 = \#\{j:\ \beta^*_j = \beta^*_{j+1}\ \&\ \hat{\beta}_j \neq \hat{\beta}_{j+1}\} \end{cases}$$

$$\text{False negatives}\begin{cases} FN1 = \#\{j:\ \beta^*_j \neq 0\ \&\ \hat{\beta}_j = 0\} \\ FN2 = \#\{j:\ \beta^*_j \neq \beta^*_{j+1}\ \&\ \hat{\beta}_j = \hat{\beta}_{j+1}\} \end{cases}$$

$$\text{True positives}\begin{cases} TP1 = \#\{j:\ \beta^*_j \neq 0\ \&\ \hat{\beta}_j \neq 0\} \\ TP2 = \#\{j:\ \beta^*_j \neq \beta^*_{j+1}\ \&\ \hat{\beta}_j \neq \hat{\beta}_{j+1}\} \end{cases}$$

$$\text{True negatives}\begin{cases} TN1 = \#\{j:\ \beta^*_j = 0\ \&\ \hat{\beta}_j = 0\} \\ TN2 = \#\{j:\ \beta^*_j = \beta^*_{j+1}\ \&\ \hat{\beta}_j = \hat{\beta}_{j+1}\} \end{cases}$$

$$\text{False discovery rate}\left\{ FDRi = \frac{FPi}{FPi + TPi} \quad i = 1, 2 \right.$$

$$\text{False negative rate}\left\{ FNRi = \frac{FNi}{FNi + TNi} \quad i = 1, 2 \right.$$

### 4.3.3 Mean squared error

Moreover, we will calculate the Mean Squared Error (MSE). The MSE of an estimator is one of many ways to quantify the difference between an estimator and the true value of the quantity being estimated. Let $\hat{\beta}_k$ is an estimator of $\beta^*_k$, $n$ is the length of the simulated $y$, then the MSE of $\hat{\beta}_k$ with respect to $\beta^*_k$ is

$$MSE(\hat{\beta}_k) = Var(\hat{\beta}_k) + (Bias(\hat{\beta}_k, \beta^*_k))^2 \quad \text{for } k = 1, \ldots, n.$$

The MSE thus assesses the quality of an estimator in terms of its variation and unbiasedness. To be more specific, let $nr$ is one of the $Nr$ simulations we have and $\hat{\beta}^{nr}_k$ is the estimator of $\beta_k^*$, therefore

$$Var(\hat{\beta}_k) = \frac{1}{Nr}\sum_{nr=1}^{Nr}(\hat{\beta}_k^{nr})^2 - \frac{1}{Nr}\left(\sum_{nr=1}^{Nr}\hat{\beta}_k^{nr}\right)^2 \text{ for } k = 1, \ldots, n.$$

And

$$(Bias(\hat{\beta}_k, \beta_k^*))^2 = \frac{1}{Nr}\sum_{nr=1}^{Nr}(\hat{\beta}_k^{nr} - \beta_k^*)^2 \text{ for } k = 1, \ldots, n.$$

So for every coefficient there is a corresponding MSE. The final MSE of the model is the mean of $n$ values calculated by the formulas above.

## 4.4 Results

We apply the two algorithms on the 500 simulations of each dataset, with the several level of noise. For each simulation, we calculate all the metrics described in the previous section, then we calculate their **means**. Moreover, we calculate the means of the optimal values of $\lambda_1$ and $\lambda_2$, we will call them $\bar{\lambda}_1$ and $\bar{\lambda}_2$. We report the results in Table 4.2 and Table 4.3.

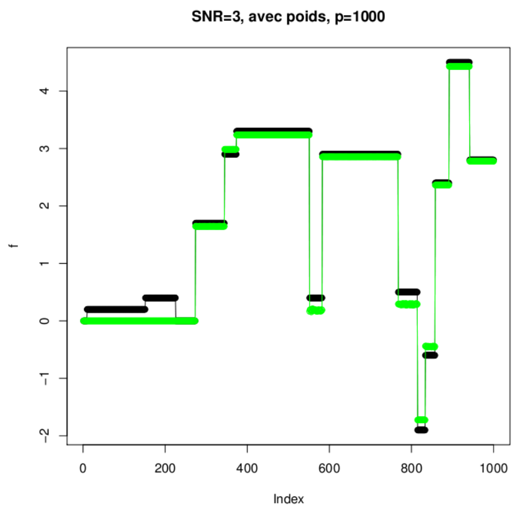
TAB. 4.2 – Comparison of the results obtained by FLSA and A-FLSA, n=100.

| | A-FLSA | | | FLSA | | |
|---|---|---|---|---|---|---|
| SNR | 7 | 5 | 3 | 7 | 5 | 3 |
| FDR 1 | 0.03483 | 0.04457 | 0.05687 | 0.06454 | 0.06647 | 0.06634 |
| FDR 2 | 0.48719 | 0.44719 | 0.48073 | 0.42865 | 0.48213 | 0.55175 |
| FNR 1 | 0.3901 | 0.04457 | 0.24545 | 0.10625 | 0.06647 | 0.15863 |
| FNR 2 | 0.02848 | 0.44719 | 0.04237 | 0.02735 | 0.48213 | 0.03863 |
| $\bar{\lambda}_1$ | 0.0642 | 0.0538 | 0.0538 | 0.0244 | 0.0268 | 0.0458 |
| $\bar{\lambda}_2$ | 0.634 | 0.782 | 1.062 | 0.5016 | 0.5404 | 0.6432 |
| MSE | 0.05323 | 0.07095 | 0.13653 | 0.03454 | 0.05153 | 0.11681 |

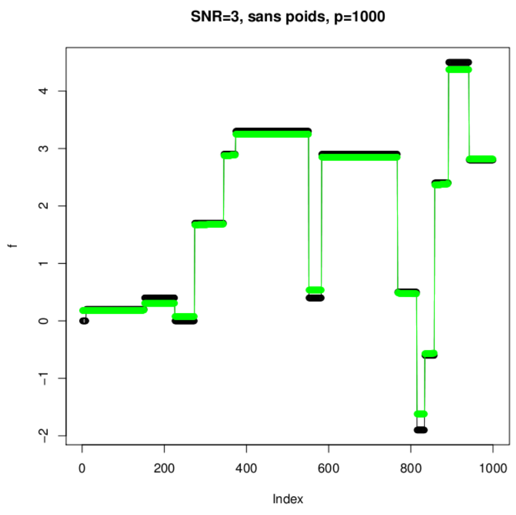
TAB. 4.3 – Comparison of the results obtained by FLSA and A-FLSA, n=1000.

| | A-FLSA | | | FLSA | | |
|---|---|---|---|---|---|---|
| SNR | 7 | 5 | 3 | 7 | 5 | 3 |
| FDR 1 | 0.05689 | 0.05504 | 0.05029 | 0.05589 | 0.05426 | 0.05195 |
| FDR 2 | 0.69368 | 0.73686 | 0.81771 | 0.83268 | 0.83230 | 0.82899 |
| FNR 1 | 0.00146 | 0.05504 | 0.11707 | 0.00301 | 0.05426 | 0.07479 |
| FNR 2 | 0.00344 | 0.73686 | 0.00542 | 0.00269 | 0.83230 | 0.00465 |
| $\bar{\lambda}_1$ | 0.0002 | 0.0038 | 0.0202 | 0.0028 | 0.0086 | 0.0226 |
| $\bar{\lambda}_2$ | 1.1432 | 1.9828 | 4.3256 | 1.1104 | 1.5324 | 2.5496 |
| MSE | 0.01173 | 0.02679 | 0.08950 | 0.01012 | 0.01824 | 0.04351 |

We show in these two figures below the original simulated values of $\beta^*$ (black) and the mean of estimated $\hat{\beta}$ (green) in the case where $n = 1000$.



(a) Using A-FLSA



(b) Using FLSA

FIG. 4.3 – Simulated $\beta^*$ and their estimates using the two methods.

## 4.5 Comparison

Looking at the two tables 4.2 and 4.3, we can say that there is a little difference between the two methods FLSA and A-FLSA, but this difference is not enough to declare that adding weights in the model return more important results. For example, FDR1 and FDR2 in the A-FLSA case are smaller than in FLSA case, while MSE is not. As for the

FNR2, we see that sometimes is better in A-FLSA, and sometines it is not.
We always believe that A-FLSA method is better than FLSA, from theoretic point of view, we suggest some reasons why we had not such big difference between the two methods :

- Since we are not sure about the hypotheses we need to propose about the second term in the fused Lasso model, it is possible that we may try another criterion than cross validation for choosing $\lambda_1$ and $\lambda_2$. For example, try to find a criterion that gives the optimal value of $\lambda_1$ first, then for this $\lambda_1$ find the optimal $\lambda_2$.
- Another suggestion is trying to adapt only the first term of the fused Lasso model, therefore adding weights only to adapt the absolute values of the coefficients.
- Another reason we suggest is that maybe by using a bigger $\gamma$ in the weights we get a better solution, in this report we used $\gamma = 1$.

We will try to explore all these suggestions in the futur.

# 5 Conclusion and perspectives

In this report, we compared two regression/segmentation methods. We added some weights in the well known fused Lasso regression, we called it adaptive fused Lasso. We adapted a path algorithm that solves the fused Lasso in the case where the design matrix $\mathbf{X}$ is the identity matrix and the number of observations is equal to the number of parameters (fused Lasso signal approximator). We concluded that those weights are not so useful in this case, they did not gave us better solutions. Probably, we are in a situation where it is useful to adapt the Lasso penalty on the absolute values of the coefficients but no need to adapt the Lasso penalty on the absolute value of their diffrence.
In the futur, we will investigate more about the criterion to choose the optimal values of $\lambda_1$ and $\lambda_2$, we will study the suggestions that we proposed in the final comparison in this report. From a theoretic point of view, we will investigate whether the adaptive fused Lasso penalty has the oracle properties in the Gaussian regression context. In particular, where we use a random matrix $\mathbf{X}$ as a design matrix. To do this, we will use as in [15] some epiconvergence results of [5]. because [15] has already found some scenarios where the Lasso does not satisty the oracle properties, while the adaptive Lasso satisfies these properties. From the practical point of view, we will try to adapt an algorithm that solves the fused Lasso with a random design matrix $\mathbf{X}$, therefore we have to find an algorithm in which theorem 2 holds. We also have to prove the asymptotical results of the adaptive fused Lasso.
Finally, we will apply our algorithm on real data.

# A Appendix

*Proof of Theorem 1.* We will use the same proof presented in [4]. By adding the weights that we use in the adaptive fused Lasso, some changes will occur.
First we find the subgradient equations for $\beta_1, \cdots, \beta_n$, which are

$$g_i = \frac{\partial L(y, \beta)}{\partial \beta_i} = -(y_i - \beta_i) + \lambda_1 w_i^{(1)} S_i + \lambda_2 w_i^{(2)} T_{i,i+1} - \lambda_2 w_{i-1}^{(2)} T_{i-1,i}$$

where

- $S_i = sign(\beta_i)$ if $\beta_i \neq 0$
  $S_i \in [-1, 1]$ if $\beta_i = 0$, it can be chosen arbitrary (see [4]).
- $T_{i,j} = sign(\beta_i - \beta_j)$ if $\beta_i \neq \beta_j$
  $T_{i,j} \in [-1, 1]$ if $\beta_i = \beta_j$

These equations are necessary and sufficient for the solution. As it assumed that a solution for $\lambda_1 = 0$ is known, let $S(0)$ and $T(0)$ denote the values of the parameters for this solution. To be more Specific, $S_i(0) = sign(\hat{\beta}_i(0))$ for $\hat{\beta}_i(0) \neq 0$. If $\hat{\beta}_i(0) = 0$ we choose $S_i(0) = 0$, since it can be chosen arbitrarily ([4]). Note that as $\lambda_2$ is constant throughout the whole proof, the dependence of $\beta, S$ and $T$ on $\lambda_2$ is suppressed for notational convenience.

In order to find $T(\lambda_1)$, observe that soft-thresholding of $\beta(0)$ does not change the ordering of pairs $\hat{\beta}_i(\lambda_1)$ and $\hat{\beta}_j(\lambda_1)$ for those pairs for which at least one of the two is not 0 and, therefore, it is possible to define $T_{i,j}(\lambda_1) = T_{i,j}(0)$. If $\hat{\beta}_i(\lambda_1) = \hat{\beta}_j(\lambda_1) = 0$, then $T_{i,j}$ can be chosen arbitrarily in [-1,1] and, therefore, let $T_{i,j}(\lambda_1) = T_{i,j}(0)$. Thus, without violating restrictions on $T_{i,j}$, we have $T(\lambda_1) = T(0)$ for all $\lambda_1 > 0$. Otherwise, $S(\lambda_1)$ will be chosen appropriately below so that the subgradient equations hold.

Now insert $\hat{\beta}_i(\lambda_1) = sign(\hat{\beta}_i(0)) \, (|\hat{\beta}_i(0)| - w_1^{(i)} \lambda_1)^+$ into the subgradient equations. For $\lambda_1 > 0$, look at 2 cases :

**Case 1** $|\hat{\beta}_i(0)| \geq w_1^{(i)}\lambda_1 \Rightarrow \hat{\beta}_i(\lambda_1) = \hat{\beta}_i(0) - w_1^{(i)}\lambda_1 S_i(0)$ . Then,

$$\begin{aligned} g_i(\lambda_1) = -y_i + \hat{\beta}_i(\lambda_1) + w_1^{(i)}\lambda_1 S_i(\lambda_1) + w_i^{(2)}\lambda_2 T_{i,i+1} - w_{i-1}^{(2)}\lambda_2 T_{i-1,i} \\ = -y_i + \hat{\beta}_i(0) - w_1^{(i)}\lambda_1 S_i(0) + w_1^{(i)}\lambda_1 S_i(\lambda_1) + w_i^{(2)}\lambda_2 T_{i,i+1} - w_{i-1}^{(2)}\lambda_2 T_{i-1,i} \\ = -y_i + \hat{\beta}_i(0) + w_i^{(2)}\lambda_2 T_{i,i+1} - w_{i-1}^{(2)}\lambda_2 T_{i-1,i} = 0 \end{aligned}$$

by setting $S_i(\lambda_1) = S_i(0)$, and using the definition of $T(\lambda_1)$ and noting that $\hat{\beta}(0)$ was assumed to be a solution.

**Case 2** $|\hat{\beta}_i(0)| < w_1^{(i)}\lambda_1 \Rightarrow \hat{\beta}_i(\lambda_1) = 0$. Then,

$$\begin{aligned} g_i(\lambda_1) = -y_i + w_1^{(i)}\lambda_1 S_i(\lambda_1) + w_i^{(2)}\lambda_2 T_{i,i+1} - w_{i-1}^{(2)}\lambda_2 T_{i-1,i} \\ = -y_i + \hat{\beta}_i(0) + w_i^{(2)}\lambda_2 T_{i,i+1} - w_{i-1}^{(2)}\lambda_2 T_{i-1,i} = 0 \end{aligned}$$

by setting $S_i(\lambda_1) = \frac{\hat{\beta}_i(0)}{w_1^{(i)}\lambda_1}$ that is $\in$ [-1,1].

$\square$

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