Mathematics for Data Science

Lecture 1: Probability theory and Discrete Random Variables

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About me

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Plan

Introduction to probability theory

Real Random Variable

Discrete Random Variables

Moments of a discrete random variable

Two Random Variables

Uniform distribution $\mathcal{U}(n)$

Bernoulli distribution $\mathcal{B}(p)$

Binomial distribution $\mathcal{B}(n, p)$

Poisson distribution $\mathcal{P}(\lambda)$

Introduction to probability theory

Randomness (Uncertainty)

Fundamental example: consider the game of a die throw.

- Fundamental example ε : "throw a balanced die" ⟵ `Action`.
- Sample space: the set of all possible results of this random experiment

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

- Events: In this random experiment, one can be interested in more complex events than just a simple result of the experiment.
- The The Power set Ω, called $\mathcal{P}(\Omega)$, is the set of all subsets of Ω.
- A family of subsets $\mathcal{A}$ of Ω. These subsets are called events. We say that the event A has occured if and only if the result ω of Ω that has occurred belongs to A.
- **σ-Algebra**: We call σ-Algebra any family $\mathcal{A}$ of subsets of Ω satisfying:
  1. $\Omega \in \mathcal{A}$.
  2. if $A \in \mathcal{A}$, then $\bar{A} \in \mathcal{A}$.
  3. if $(A_n)_{n\in\mathbb{N}}$ is a sequence of elements in $\mathcal{A}$, then $\bigcup\limits_{n\in\mathbb{N}} A_n \in \mathcal{A}$.
- $(\Omega, \mathcal{A})$ is a measurable space (or a Borel space).

Notions on Events

- Let $(\Omega, \mathcal{A})$ be a measurable space:
    - The set $\mathcal{A}$ is called σ-Algebra of events. The elements of $\mathcal{A}$ are called the events.
    - The event Ω is called certain event. The event $\emptyset$ is called impossible event.
- Operations on events. Let A and B be two events:
    - $\bar{A}$ is the complement event of A (we also note A^c). $\bar{A} = \Omega \setminus A$.
      $barA$ occurs if and only if A does not occur.
    - $A{\cap}B$ is the event «A and B».
      $A \cap B$ occurs when both events occur.
    - $A \cup B$ is the event «A or B».
      $A \cup B$ occurs when at least one of the two events occurs.
- Mutually Exclusive Events: A and B are mutually exclusive if their simultaneous realization is impossible: $A \cap B = \emptyset$.
- Implication: A implies B means that if A occurs, then B also occurs: $A \subset B$.

Probability Space

- Let $(\Omega, \mathcal{A})$ a measurable space. A probability function on $(\Omega, \mathcal{A})$, is any map

$$P : \mathcal{A} \to \mathbb{R}$$

such that:

1. $\forall A \in \mathcal{A}, P(A) \geqslant 0$.
2. $P(\Omega) = 1$.
3. $\forall (A_n)_{n \in \mathbb{N}^*} \in \mathcal{A}^{\mathbb{N}^*}$, a familty of pairwise disjoint (mutually exclusive) events, we have:

$$P(\bigcup_{n \in \mathbb{N}^*} A_n) = \sum_{n=1}^{+\infty} P(A_n)$$

- The triplet $(\Omega, \mathcal{A}, P)$ is called a **probability space**.

Probability: Properties

1. $P(\emptyset) = 0$.
2. $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$.
3. If A_1 and A_2 are mutually exclusive, $A_1 \cap A_2 = \emptyset$, $P(A_1 \cup A_2) = P(A_1) + P(A_2)$.
4. $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$.
5. $P(\bar{A}) = 1 - P(A)$.
6. $P(B \setminus A) = P(B) - P(B \cap A)$.
7. $A \subset B \Rightarrow P(A) \leqslant P(B)$.

Uniform probability on finite Ω

- Let Ω be a finite sample space. We say that P is the **uniform probability** on the measurable space $(\Omega, P(\Omega))$ if:

$$\forall \omega, \omega' \in \Omega, \qquad P(\omega'\}) = P(\omega'\})$$

  One also says that there is **equiprobability** of elementary events.

- Let $(\Omega, \mathcal{P}(\Omega), P)$ be a finite probability space. If P is the uniform probability, then

$$\forall A \in \mathcal{A}, \qquad P(A) = \frac{\mathrm{Card}(A)}{\mathrm{Card}(\Omega)}$$

Conditional probability

- Let $(\Omega, \mathcal{A}, P)$ be a probability space and $B \in \mathcal{A}$ such that $P(B) > 0$. The map function P_B defined on $\mathcal{A}$ by:

$$P_B(A) = P(A|B) = \frac{P(A \cap B)}{P(B)}, \qquad \forall A \in \mathcal{A}$$

  is a probability function on $(\Omega, \mathcal{A})$; it is called the conditional probability given B. It is the probability of event A occurring given that event B has occurred.

- Remark: $(A|B)$ is not an event! We use the notation $P(A|B)$ for simplicity, but the notation $P_B(A)$ is the correct one.

- Chain rule:

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

- Law of total probability:
  - $\forall A \in \mathcal{A}, \quad P(A) = P(A \cap B) + P(A \cap \bar{B})$
  - We call partition of Ω, a set of events that are pairwise disjoint and whose union is the sample space Ω. The partition is said to be “countable” if its cardinality is at most equal to that of $\mathbb{N}$.
  - Let $(B_n)_{n \geqslant 0}$ a partition of Ω. We have:

$$\forall A \in \mathcal{A}, \qquad P(A) = \sum_{n \geqslant 0} P(A \cap B_n)$$

- Independence: Events A and B are independent iff $P(A \cap B) = P(A)P(B)$.

Bayes' formulae

First Bayes' theorem

Let $(\Omega, \mathcal{A}, P)$ a probability space. For all events A and B such that $P(A) \neq 0$ and $P(B) \neq 0$, we have:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Second Bayes' theorem

Let $(\Omega, \mathcal{A}, P)$ a probability space and $(B_n)_{n\geqslant 0}$ a partition of Ω s.t. for all $n \geqslant 0$ $P(B_n) \neq 0$. We have for all $A \in \mathcal{A}$ s.t. $P(A) \neq 0$

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{n\geqslant 0} P(A|B_n)P(B_n)} \qquad \forall i \geqslant 0$$

Real Random Variable

Concept of Real Random Variable

Definition

Let ε an experiment and $(\Omega, \mathcal{A}, \mathrm{P})$ the associated probability space. . In many situations, one associates to each result $\omega \in \Omega$ a real number denoted $\mathrm{X}(\omega)$; In this way, one builds a map $\mathrm{X} : \Omega \to \mathbb{R}$. Historically, ε was a game and X représented the earning of a player.

Example: a die throw

A player throws a fair six faces dice and we observe the obtained number:

- If the result is 1,3 or 5, the player earns 1 euro.
- If the result is 2 or 4, the player earns 5 euros.
- If the result is 6, the player loses 10 euros.

Concept of Real Random Variable

Analysis

- ε: "throw a fair die".
- $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- $\mathcal{A} = \mathcal{P}(\Omega)$.
- P is the equiprobability on $(\Omega, \mathcal{A})$.

Let X the map function from Ω to $\mathbb{R}$ that associates to each result the corresponding earning.

So we have

- $X(1) = X(3) = X(5) = 1$
- $X(2) = X(4) = 5$
- $X(6) = -10$

We say that X is a **random variable** on Ω.

Concept of Real Random Variable

One can ask what is the probability for the player to win 1 euro:

⇒ $X(\omega) = 1$.

- this is the case if and only if $\omega \in \{1, 3, 5\}$.
- The sought-for probability is therefore $P(\{1, 3, 5\}) = 1/2$.
- Which can also be written as $P(X = 1) = 1/2$.

Thus, we will consider the event:

$$\{X = 1\} = \{\omega \in \Omega / X(\omega) = 1\} = \{\omega \in \Omega / X(\omega) \in \{1\}\} = X^{-1}(\{1\}) = \{1, 3, 5\}.$$

Similarly, we have:

- $P(X = 5) = 1/3$.
- $P(X = -10) = 1/6$.

Concept of Real Random Variable

One can present the three previous probabilities in a table:

x_i	-10	1	5
$p_i = P(X = x_i)$	1/6	1/2	1/3

This is tantamount of considering a new sample space:

$$\Omega_X = X(\Omega) = \{-10, 1, 5\}$$

equipped with the probability P_X defined in the table above. This new probability is called the **probability distribution** of X.

Notice that

$$P(\bigcup_{x_i \in \Omega_X} \{X = x_i\}) = \sum_{x_i \in \Omega_X} P(X = x_i) = 1$$

Plan: Discrete Random Variables

In this chapter:

- We treat the case where $X(\Omega)$ is countable.
- The random variable in this case is ***discrete***.
- We define its probability law by its probability distribution.
- We will define the two main numerical characteristics of a discrete random variable:
  - Expected value: characteristic of centrality (the *mean*).
  - Variance: characteristic of dispersion.
- We will also define the couples of random variables.

Discrete Random Variables

Probability law

Definition

*We say that a real random variable X is **discrete** if the set of all possible values that X can take is finite or countable.*

If we suppose that the set $X(\Omega)$ of all possible values of X admits a smallest element x_1. Then the discrete real random variable X is completely defined by:

- *The set $X(\Omega)$ of all possible values of X, sorted in ascending order: $X(\Omega) = \{x_1, x_2, \dots, x_i, \dots\}$ with $x_1 \leqslant x_2 \leqslant \dots \leqslant x_i \leqslant \dots$.*
- *The probability distribution defined on $X(\Omega)$ by*

$$p_i = p(x_i) = P(X = x_i) \quad \forall\, i = 1, 2, \dots$$

Remarks:

- $B \subset \mathbb{R}$, $P(X \in B) = \sum_{i/x_i \in B} p(x_i)$.
- $P(a < X \leqslant b) = \sum_{i/a < x_i \leqslant b} p(x_i)$.
- $p(x_i) \geqslant 0$ and $\sum_{i=1}^{\infty} p(x_i) = 1$.
- If the number of possible values of X is small enough, the probability distribution of X is often presented as a table.

Cumulative distribution function

Definition

Given a discrete random variable X, we call cumulative distribution function of X (or simply distribution function), denoted F_X, the function defined by: for any real a,

$$F(a) = P(X \leqslant a) = \sum_{i/x_i \leqslant a} P(X = x_i)$$

The value $F_X(a)$ represents the probability that X takes a value smaller or equal to a.

Properties

1. It is a staircase function.
2. $F(a) \leqslant 1$ since it is a probability.
3. $F(a)$ is continuous from the right.
4. $\lim\limits_{a \to -\infty} F(a) = 0$ and $\lim\limits_{a \to \infty} F(a) = 1$

The distribution function characterizes the distribution of X. In other words, if X and Y are two random variables, we have $F_X = F_Y$ if and only if their probability distributions are the same.

Distribution function and probabilities over X

All the computations of probabilities about X can be carried out using the distribution function. For example,

$$P(a < X \leqslant b) = F(b) - F(a) \qquad \text{pour tout } a < b$$

This is easier to understand if one writes the event $\{X \leqslant b\}$ as a union of two incompatible events $\{X \leqslant a\}$ and $\{a < X \leqslant b\}$, Let

$$\{X \leqslant b\} = \{X \leqslant a\} \cup \{a < X \leqslant b\}$$

In this way,

$$P(X \leqslant b) = P(X \leqslant a) + P(a < X \leqslant b)$$

which proves the equality above.

Remark

One can compute the individual probabilities by:

$$p_i = P(X = x_i) = F(x_i) - F(x_{i-1}) \qquad \text{pour } 1 \leqslant i \leqslant n$$

Distribution function and probabilities over X

Example

We play three times to “heads or tails” $\Rightarrow$

- $\Omega = \{P, F\}^3$.
- $card(\Omega) = |\Omega| = 2^3 = 8$.

Let X the random variable “number of tails obtained” $\Rightarrow$ $X(\Omega) = \{0, 1, 2, 3\}$.

- Let’s calculate $P(X = 1)$.
- $X^{-1}(1) = \{(P, F, F), (F, P, F), (F, F, P)\}$.

$\Rightarrow$ $P(X = 1) = \frac{3}{8}$

Using the same method we obtain the probability distribution of X:

k	0	1	2	3
$P(X = k)$	1/8	3/8	3/8	1/8

Distribution function and probabilities over X

The distribution function X is therefore given by:

$$F(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1/8 & \text{si } 0 \leqslant x < 1 \\ 1/2 & \text{si } 1 \leqslant x < 2 \\ 7/8 & \text{si } 2 \leqslant x < 3 \\ 1 & \text{si } x \geqslant 3 \end{cases}$$

One can represent both the probability distribution and the distribution function of X in the same table:

k	0	1	2	3
$P(X = k)$	$1/8$	$3/8$	$3/8$	$1/8$
$F_X(x)$	$1/8$	$1/2$	$7/8$	1

Distribution function and probabilities over X

The graph of the distribution function is represented below:

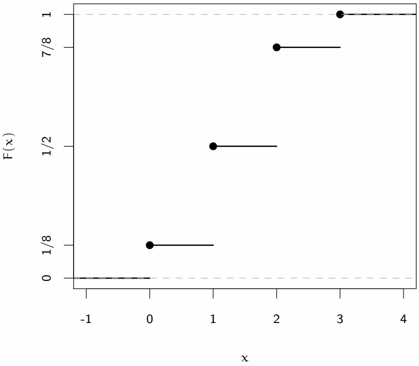


Figure 1: Distribution function

Distribution function and probabilities over X

Here is another slightly different representation of the distribution function:

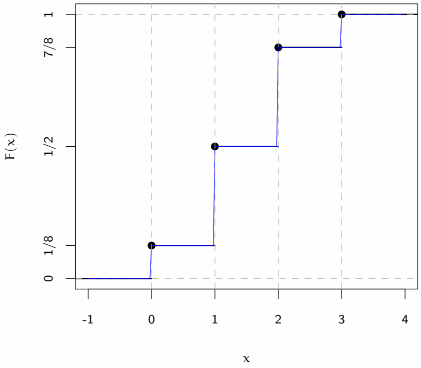


Figure 2: Distribution function

The indicator random variable

Definition

Let A an event. We call indicator random variable of the event A, the random variable $X = \mathbb{1}_A$ defined by:

$$X(\omega) = \begin{cases} 1 & \textit{si } \omega \in A \\ 0 & \textit{si } \omega \in \bar{A} \end{cases}$$

Therefore:

- $P(X = 1) = P(A) = p$
- $P(X = 0) = P(\bar{A}) = 1 - p$

The distribution function of the indicator random variable is therefore:

$$F(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1 - p & \text{si } 0 \leqslant x < 1 \\ 1 & \text{si } x \geqslant 1 \end{cases}$$

The indicator random variable: Example

Example

- Let $\mathcal{U}$ an urn containing two white ball and three red balls.
- We randomly take one ball out of the box.
- Let A : “take one white ball out”.
- Let X be the indicator random variable of A.

Find the probability distribution and the distribution function of X.

The probability distribution of X is

k	0	1
$P(X=k)$	$\frac{3}{5}$	$\frac{2}{5}$

and its distribution function is:

$$F(x) = \begin{cases} 0 & \text{si } x < 0 \\ 3/5 & \text{si } 0 \leqslant x < 1 \\ 1 & \text{si } x \geqslant 1 \end{cases}$$

Moments of a discrete random variable

Expected value

Definition

For a discrete random variable X *with probability distribution* $p(.)$, *we define the expected value of* X, *called* $E(X)$, *by*

$$E(X) = \sum_{i \in \mathbb{N}} x_i p(x_i)$$

In concrete terms, the expected value of X *is the weighted mean of the values of* X, *the weights being the probabilities associated to the values of* X.

Examples

1. In the previous example where we play three times to “heads or tails”, the expected value of X is:

$$E(X) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = 1.5$$

2. For the indicator random variable of A:

$$E(X) = 0 \times P(X = 0) + 1 \times P(X = 1) = P(A) = p$$

which means that the expected value of the indicator of an event A corresponds to the probability that A occurs.

Expected value of a function of a random variable

Theorem

Let X be a discrete random variable whose possible values are x_i, $i \geqslant 1$, and denote by $p(x_i)$ the probability that $X = x_i$ occurs. Then, for any real function g, we have

$$E(g(X)) = \sum_i g(x_i)p(x_i)$$

Example

Let X be a random variable that can take three values $\{-1, 0, 1\}$ with the following probabilities:

$$P(X = -1) = 0.2 \qquad P(X = 0) = 0.5 \qquad P(X = 1) = 0.3$$

Calculate $E(X^2)$.

Expected value of a function of a random variable

Solution

First method: Let $Y = X^2$. The probability distribution of Y is given by

$$P(Y=1) = P(X=-1) + P(X=1) = 0.5$$
$$P(Y=0) = P(X=0) = 0.5$$

So

$$E(X^2) = E(Y) = 1(0.5) + 0(0.5) = 0.5$$

Second method: Using the theorem

$$\begin{aligned} E(X^2) &= (-1)^2(0.2) + 0^2(0.5) + 1^2(0.3) \\ &= 1(0.2+0.3) + 0(0.5) = 0.5 \end{aligned}$$

Remark

$$0.5 = E(X^2) \neq (E(X))^2 = 0.01$$

Linearity of the expected value

Properties

1. $E(X+a) = E(X) + a, \quad a \in \mathbb{R}$
   results which follows from:

$$\sum_i p_i(x_i + a) = \sum_i p_i x_i + \sum_i a p_i = \sum_i p_i x_i + a \sum_i p_i = \sum_i p_i x_i + a$$

2. $E(aX) = aE(X), \quad a \in \mathbb{R}$
   to prove it, just write:

$$\sum_i p_i a x_i = a \sum_i p_i x_i$$

3. $E(X+Y) = E(X) + E(Y)$, X and Y being two random variables.

All these three properties are summarised in the claim that the expected value is linear:

$$E(\lambda X + \mu Y) = \lambda E(X) + \mu E(Y), \quad \forall \lambda \in \mathbb{R}, \forall \mu \in \mathbb{R}.$$

Variance

Definition

Let X be a discrete random variable. We call variance of X, denoted $V(X)$, the quantity defined by, when it exists,

$$V(X) = E\left[(X - E(X))^2\right]$$

Thus, the variance is the expected value of the square of the centered random variable $X - E(X)$. The variance can be interpreted as a measure of the dispersion of the possible values of X around its expected value.

Remark

Equivalently, the variance might be defined by the following formula:

$$V(X) = E(X^2) - E^2(X)$$

Indeed:

$$\begin{aligned} V(X) &= E\left[X^2 - 2XE(X) + E^2(X)\right] \\ &= E(X^2) - E[2XE(X)] + E[E^2(X)] \\ &= E(X^2) - 2E^2(X) + E^2(X) \end{aligned}$$

Variance

Example

OLet us compute $V(X)$ in the case where X is the number obtained when throwing a fair die.

Previously, we saw that $E(X) = \frac{7}{2}$. Moreover,

$$
\begin{aligned}
E(X^2) &= \sum_i x_i^2 p(x_i) \\
&= 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{6}\right) + 3^2\left(\frac{1}{6}\right) + 4^2\left(\frac{1}{6}\right) + 5^2\left(\frac{1}{6}\right) + 6^2\left(\frac{1}{6}\right) \\
&= \left(\frac{1}{6}\right)(91) = \frac{91}{6}
\end{aligned}
$$

And therefore

$$
\begin{aligned}
V(X) &= E(X^2) - E^2(X) \\
&= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}
\end{aligned}
$$

Variance

Properties

1. $V(X) \geqslant 0$

2. $\forall\, a \in \mathbb{R}, \quad V(X+a) = V(X)$

   en effet:

$$\begin{aligned} V(X+a) &= E\left[[X+a-E(X+a)]^2 \right] \\ &= E\left[[X+a-E(X)-a]^2 \right] \\ &= E\left[[X-E(X)]^2 \right] = V(X) \end{aligned}$$

3. $\forall\, a \in \mathbb{R}, \quad V(aX) = a^2 V(X)$

   en effet:

$$\begin{aligned} V(aX) &= E\left[[aX-E(aX)]^2 \right] \\ &= E\left[[aX-aE(X)]^2 \right] \\ &= E\left[a^2 [X-E(X)]^2 \right] \\ &= a^2 \left[E\, [X-E(X)]^2 \right] = a^2 V(X) \end{aligned}$$

Standard deviation

Definition

Let X be a discrete random variable. The square root of the variance is called the ***standard deviation*** *of X and is denoted*

$$\sigma_X = \sqrt{V(X)}$$

σ_X has the same physical units as the random variable X.

- The standard deviation allows to measure the dispersion of a set of data.
- The smaller sigma is, the closer to each other the values of the data are.
- Example: the dispersion of the grades in an exam. The smaller sigma is, the more homogeneous the class is.
- - Expected value and standard deviation are linked through *Bienaymé-Tchebychev inequality*.

Bienaymé-Tchebychev inequality

Theorem

Let X *a random variable of expected value* μ *and variance* σ^2. *For all* $\varepsilon > 0$, *We have:*

$$P\left(|X - E(X)| \geqslant \varepsilon\right) \leqslant \frac{\sigma^2}{\varepsilon^2}$$

Remark

This inequality can be written in a slightly different fashion. Let $k = \varepsilon/\sigma$.

$$P\left(|X - E(X)| \geqslant k\sigma\right) \leqslant \frac{1}{k^2}$$

Importance

This inequality relates the probability for X to deviate from its expected value $E(X)$ to its variance, which is precisely an indicator of the dispersion around the expected value. The inequality makes quantitatively precise the statement "the smaller the variance is, the less likely it is to find values far away from the expected value".

Centred and non-centred moments

Definition

We call non centered moment of order $r \in \mathbb{N}^*$ *of* X *the quantity, when it exists:*

$$m_r(X) = \sum_{i \in \mathbb{N}} x_i^r p(x_i) = E(X^r).$$

Definition

The centered moment of order $r \in \mathbb{N}^*$ *the quantity, when it exists:*

$$\mu_r(X) = \sum_{i \in \mathbb{N}} p_i \left[x_i - E(X)\right]^r = E\left[X - E(X)\right]^r.$$

Remark

The first moments are:

- $m_1(X) = E(X), \quad \mu_1(X) = 0.$
- $\mu_2(X) = V(X) = m_2(X) - m_1^2(X).$

Two Random Variables

Two Discrete Random Variables

So far, we have dealt with one random variable. However, it is often necessary to consider events related to two variables simultaneously, or even to more than two variables.

Definition

Let X and Y two discrete random variables, defined on probability space $(\Omega, \mathcal{A}, P)$ and that $X(\Omega) = \{x_1, x_2, \dots, x_l\}$ and $Y(\Omega) = \{y_1, y_2, \dots, y_k\}$, l and $k \in \mathbb{N}$.

*The **probability law of** (X, Y) is defined by joint probabilities:*

$$p_{ij} = P(X = x_i; Y = y_j) = P(\{X = x_i\} \cap \{Y = y_j\})$$

We have

$$p_{ij} \geqslant 0 \quad \textit{et} \quad \sum_{i=1}^{l} \sum_{j=1}^{k} p_{ij} = 1$$

The pair (X, Y) is called two dimensional random vector and can have $l \times k$ valeurs.

Joint probability distribution table

The probabilities p_{ij} can be presented in a two dimensional table than we call joint probability distribution table:

Table 1: Joint probability distribution table

$X \backslash Y$	y_1	y_2	$\dots$	y_j	$\dots$	y_k
x_1	p_{11}	p_{12}		p_{1j}		p_{1k}
x_2	p_{21}	p_{22}		p_{2j}		p_{2k}
$\vdots$						
x_i	p_{i1}	p_{i2}		p_{ij}		p_{ik}
$\vdots$						
x_l	p_{l1}	p_{l2}		p_{lj}		p_{lk}

In the header we have the possible values of Y and in the first column the possible values of X. The probability $p_{ij} = P(X = x_i; Y = y_j)$ is at the intersection of i^{th} line and j^{th} column.

Example: pair of random variables

Example

Three balls are drawn at random from an urn containing 3 red, 4 white and 5 black balls. X and Y are respectively the number of red and white balls drawn. Determine the joint probability distribution of the pair (X, Y).

Solution

- ε: “draw 3 balls from an urn containing 12 balls”.
- $|\Omega| = C_{12}^{3} = 220$.
- $X(\Omega) = \{0, 1, 2, 3\}$ and $Y(\Omega) = \{0, 1, 2, 3\}$.
- $p(X = 0, Y = 0) = p(0, 0) = C_5^3 / C_{12}^3 = \frac{10}{220}$.
- $p(0, 1) = C_4^1 C_5^2 / C_{12}^3 = \frac{40}{220}$.
- $p(1, 0) = C_3^1 C_5^2 / C_{12}^3 = \frac{30}{220}$.

Example: pair of random variables

Example

Three balls are drawn at random from an urn containing 3 red, 4 white and 5 black balls. X and Y are respectively the number of red and white balls drawn. Determine the joint probability distribution of the pair (X, Y).

Solution

Table 2: Joint probability distribution table

$X\backslash Y$	0	1	2	3
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0
3	$\frac{1}{220}$	0	0	0

Marginal Probability Distributions

When we know the joint distribution of the random variables X and Y, we can also look at the probability distribution of X alone and Y alone. These are the marginal probability distributions.

- Marginal distribution of X:

$$p_{i.} = P(X = x_i) = P[\{X = x_i\} \cap \Omega] = \sum_{j=1}^{k} p_{ij} \qquad \forall\, i = 1, 2, \ldots, l$$

- Marginal distribution of Y:

$$p_{.j} = P(Y = y_j) = P[\Omega \cap \{Y = y_j\}] = \sum_{i=1}^{l} p_{ij} \qquad \forall\, j = 1, 2, \ldots, k$$

We can calculate the marginal distributions directly from the table of the joint distribution.

Marginal Probability Distributions

Table 3: Joint distribution table with marginal distributions

$X\backslash Y$	y_1	y_2	$\dots$	y_j	$\dots$	y_k	Marginal of X
x_1	p_{11}	p_{12}		p_{1j}		p_{1k}	$p_{1.}$
x_2	p_{21}	p_{22}		p_{2j}		p_{2k}	$p_{2.}$
$\vdots$							
x_i	p_{i1}	p_{i2}		p_{ij}		p_{ik}	$p_{i.}$
$\vdots$							
x_l	p_{l1}	p_{l2}		p_{lj}		p_{lk}	$p_{l.}$
Marginal of Y	$p_{.1}$	$p_{.2}$		$p_{.l}$		$p_{.k}$	1

Example: Marginal Probability Distributions

Example

Three balls are drawn at random from an urn containing 3 red, 4 white and 5 black balls. X and Y are respectively the number of red and white balls drawn. Determine the joint probability distribution of the pair (X, Y).

Solution

Table 4: Joint distribution table

$X \backslash Y$	0	1	2	3	$p_{i.} = P(X = x_i)$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
$p_{.j} = P(Y = y_j)$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	1

Conditional Probability

Definition

For each value y_j *of* Y *such that* $p_{.j} = P(Y = y_j) \neq 0$ *we can define the conditional distribution of* X *given* $Y = y_j$ *by*

$$p_{i/j} = P(X = x_i / Y = y_j) = \frac{P(X = x_i ; Y = y_j)}{P(Y = y_j)} = \frac{p_{ij}}{p_{.j}} \qquad \forall i = 1, 2, \ldots, l$$

Same for Y *given* $X = x_i$*:*

$$p_{j/i} = P(Y = y_j / X = x_i) = \frac{P(X = x_i ; Y = y_j)}{P(X = x_i)} = \frac{p_{ij}}{p_{i.}} \qquad \forall j = 1, 2, \ldots, k$$

Independence of random variables

Definition

We say that two random variables are independent iff

$$P(X = x_i; Y = y_j) = P(X = x_i)P(Y = y_j) \qquad \forall\, i = 1, 2, \dots, l \textit{ and } j = 1, 2, \dots, k$$

One demonstrates that

$$P(\{X \in A\} \cap \{Y \in B\}) = P(\{X \in A\})P(\{Y \in B\}) \qquad \forall\ A \textit{ and } B \in \mathcal{A}$$

Properties

Let two random variables X and Y,

1. $E(X + Y) = E(X) + E(Y)$
2. If X and Y are independent so $E(XY) = E(X)E(Y)$. But the reciprocal is not always true.

Covariance

Definition

Let two random variables X and Y. The ***covariance*** *of X and Y, when it exists, is*

$$\mathrm{Cov}(X,Y) = E[(X - E(X))(Y - E(Y))] = \sum_i \sum_j (x_i - E(X))(y_j - E(Y))p_{ij}$$

that we can calculate using the formula

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

Properties

- $\mathrm{Cov}(X,Y) = \mathrm{Cov}(Y,X)$
- $\mathrm{Cov}(aX_1 + bX_2, Y) = a\mathrm{Cov}(X_1,Y) + b\mathrm{Cov}(X_2,Y)$
- $V(X+Y) = V(X) + V(Y) + 2\mathrm{Cov}(X,Y)$
- If X and Y are independant so
  - $\mathrm{Cov}(X,Y) = 0$ (the reciprocal is not always true)
  - $V(X+Y) = V(X) + V(Y)$ (the reciprocal is not always true)

Linear Correlation (Correlation of Spearman)

Definition

The correlation between X and Y is defined by

$$\rho = \rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

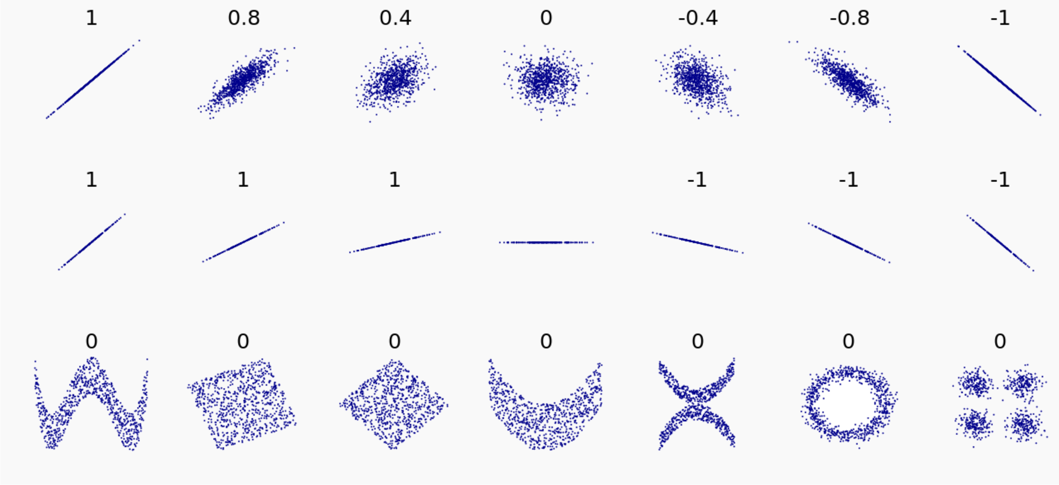
We can demonstrate that

$$-1 \leqslant \rho(X,Y) \leqslant 1$$

Interpretation of ρ

- The correlation coefficient is a measure of the degree of linearity between X and Y.
- Values of rho close to 1 or -1 indicate an almost rigorous linearity between X and Y.
- Values of rho close to 0 indicate the absence of any linear relationship.
- When $\rho(X,Y)$ is positive, Y tends to increase if X does the same.
- When $\rho(X,Y) < 0$, Y tends to decrease if X increases.

Linear correlation



Uniform distribution $\mathcal{U}(n)$

Uniform distribution

Definition

A random variable X *has a* ***discrete uniform distribution*** *if each of the* n *values in its range, say,* $x_1, x_2, \dots, x_n$, *has equal probability. Then:*

$$P(X = x_i) = \frac{1}{n} \qquad \forall i \in \{1, \dots, n\}$$

We say $X \sim \mathcal{U}(n)$.

Example

The distribution of the numbers obtained at the throw of the dice (if it is fair) follows a uniform distribution whose probability distribution is the following:

x_i	1	2	3	4	5	6
$P(X = x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Moments of uniform distribution

Particular case

In the particular case of a discrete uniform distribution where each value of the random variable X corresponds to its rank, i.e. $x_i = i \ \forall i \in \{1, \dots, n\}$, we have:

$$E(X) = \frac{n+1}{2} \quad \text{et} \quad V(X) = \frac{n^2-1}{12}$$

Demonstration

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{et} \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

Example

The example of the throw of the dice: we can directly calculate the moments of X:

$$E(X) = \frac{6+1}{2} = 3.5 \quad \text{et} \quad V(X) = \frac{6^2-1}{12} = \frac{35}{12} \simeq 2.92.$$

Bernoulli distribution $\mathcal{B}(p)$

Bernoulli distribution $\mathcal{B}(p)$

Indicator random variable

Let A an event; the indicator random variable of A, defined by $X = \mathbb{1}_A$, is:

$$X(\omega) = \mathbb{1}_A(\omega) = \begin{cases} 0 & \text{si } \omega \in \bar{A} \\ 1 & \text{si } \omega \in A \end{cases}$$

So $X(\Omega) = \{0, 1\}$ with:

$$\begin{aligned} &P(X = 1) = P\{\omega \in \Omega / X(\omega) = 1\} = P(A) = p \\ &P(X = 0) = P\{\omega \in \Omega / X(\omega) = 0\} = P(\bar{A}) = 1 - P(A) = q \\ &\text{avec } p + q = 1 \end{aligned}$$

Definition

*We say X follows a Bernoulli distribution of parameter $p = P(A)$, we write $X \sim \mathcal{B}(p)$. A Bernoulli distribution is associated to “**Bernoulli's event**”, which is a random experience having two possibilities: **success** ($X = 1$) or **fail** ($X = 0$).*

Bernoulli distribution $\mathcal{B}(p)$

Bernoulli's Distribution function

$$F(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1 - p & \text{si } 0 \leqslant x < 1 \\ 1 & \text{si } x \geqslant 1. \end{cases}$$

Expected value

$$E(X) = 1 \times P(A) + 0 \times P(\bar{A}) = P(A) = p$$

Variance

$$V(X) = E(X^2) - E^2(X) = p - p^2 = p(1-p) = pq$$

because

$$E(X^2) = 1^2 \times P(A) + 0^2 \times P(\bar{A}) = P(A) = p$$

Binomial distribution $\mathcal{B}(n, p)$

Binomial distribution $\mathcal{B}(n, p)$

- Described for the 1st time by *Isaac Newton* in 1676 and demonstrated for the 1st time by the swiss mathematician *Jacob Bernoulli* in 1713.
- Binomial distribution is one most frequently used probability distributions in applied statistics.
- n **independant** Bernoulli events.
- Each has p as probability of success and $1 - p$ probability of fail.

$$A \quad A \quad \bar{A} \quad A \quad \bar{A} \quad \dots \quad \bar{A} \quad A \quad A$$

$$S \quad S \quad E \quad S \quad E \quad \dots \quad E \quad S \quad S$$

- $X =$ **the number of success** on all n events.
- X depends of two parameters n and p.

Binomial distribution $\mathcal{B}(n, p)$

$$S \quad S \quad E \quad S \quad E \quad \ldots \quad E \quad S \quad S$$

- $X =$ **the number of success** on all n events.
- $X(\Omega) = \{0, 1, \ldots, n\}$

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad 0 \leqslant k \leqslant n$$

- $\binom{n}{k}$ is the number of all samples of size n containing exactly k successes, of probability p^k, order is not counted, and $n - k$ fails, of probability $(1-p)^{n-k}$.
- We write $X \sim \mathcal{B}(n, p)$.

Remark

A Birnoulli random variable is a Binomal variable of parameters $(1, p)$.

$$X \sim \mathcal{B}(p) \iff X \sim \mathcal{B}(1, p)$$

Pascal's triangle & Binomial theorm

Pascal's triangle

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad \forall n \geqslant 1 \text{ et } 1 \leqslant k \leqslant n-1$$

$$\begin{array}{lccccccccc}
n=0: & & & & & 1 & & & & \\
n=1: & & & & 1 & & 1 & & & \\
n=2: & & & 1 & & 2 & & 1 & & \\
n=3: & & 1 & & 3 & & 3 & & 1 & \\
n=4: & 1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

Binomial theorm

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\sum_{k=0}^{n} P(X=k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = [p + (1-p)]^n = 1$$

Binomial distribution $\mathcal{B}(n, p)$

Example

We flip five coins. The results are supposed to be independent. What is the probability distribution of X who is thenumber of heads.

Solution

- $X =$ nombre de piles (*succès*).
- $n = 5$.
- $p = 1/2$.
- $X \sim \mathcal{B}(5, \frac{1}{2})$.
- $X(\Omega) = \{0, 1, \ldots, 5\}$
- $P(X = 0) = \binom{5}{0}\left(\frac{1}{2}\right)^0\left(1 - \frac{1}{2}\right)^{5-0} = \frac{1}{32}$
- $P(X = 1) = \binom{5}{1}\left(\frac{1}{2}\right)^1\left(1 - \frac{1}{2}\right)^{4} = \frac{5}{32}$
- $P(X = 2) = \binom{5}{2}\left(\frac{1}{2}\right)^2\left(1 - \frac{1}{2}\right)^{3} = \frac{10}{32}$
- $P(X = 3) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(1 - \frac{1}{2}\right)^{2} = \frac{10}{32}$
- $P(X = 4) = \binom{5}{4}\left(\frac{1}{2}\right)^4\left(1 - \frac{1}{2}\right)^{1} = \frac{5}{32}$
- $P(X = 5) = \binom{5}{5}\left(\frac{1}{2}\right)^5\left(1 - \frac{1}{2}\right)^{0} = \frac{1}{32}$

Moments of Binomial distribution

If $X \sim \mathcal{B}(n, p)$ so $E(X) = np$ and $V(X) = np(1-p)$

demonstration

1st method: We assigne to each i, $1 \leqslant i \leqslant n$, a Bernoulli random variable

$$\mathbb{1}_A = X_i = \begin{cases} 1 & \text{if } A \text{ is realized} \\ 0 & \text{if not} \end{cases}$$

So we write: $X = \sum_{i=1}^{n} X_i = X_1 + X_2 + \ldots + X_n$

Then

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = np$$

et

$$V(X) = V\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} V(X_i) = np(1-p)$$

because X_i are indenpendent.

Moments of Binomial distribution

2nd method: Direct calculation.

- $E(X) = \sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k} = \ldots = np$
- $V(X) = E(X^2) - E^2(X)$
- To get $E(X^2)$ we go through $E[X(X-1)]$.
- $V(X) = E(X^2) - E^2(X) = E[X(X-1)] + E(X) - E(X^2)$
- $E[X(X-1)] = \sum_{k=0}^{n} k(k-1)\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k} = \ldots = n(n-1)p^2$
- $V(X) = n(n-1)p^2 + np - (np)^2 = np(1-p)$

Binomial distribution $\mathcal{B}(n, p)$

Example

The number of heads after n coin flips follows a Binomial distribution $\mathcal{B}\left(n, 1/2\right)$:

$$P(X = k) = \binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} = \frac{\binom{n}{k}}{2^n}, \quad 0 \leqslant k \leqslant n$$

with $E(X) = n/2$ and $V(X) = n/4$.

Example

The number N of red balls appearing in n draws with replacement from an urn containing two red, three green and one black follows a Binomial distribution $\mathcal{B}\left(n, 1/3\right)$:

$$P(N = k) = \binom{n}{k}\left(\frac{1}{3}\right)^k\left(\frac{2}{3}\right)^{n-k} = \binom{n}{k}\frac{2^{n-k}}{3^n}, \quad 0 \leqslant k \leqslant n$$

with $E(X) = n/3$ and $V(X) = 2n/9$.

Remark

If $X_1 \sim \mathcal{B}\left(n_1, p\right)$ et $X_2 \sim \mathcal{B}\left(n_2, p\right)$, X_1 and X_2 being **indenpendent**, so $X_1 + X_2 \sim \mathcal{B}\left(n_1 + n_2, p\right)$. This results from the definition of a Binomial distribution since we sum up here the result of $n_1 + n_2$ independent events.

Poisson distribution $\mathcal{P}(\lambda)$

Poisson distribution $\mathcal{P}(\lambda)$

Definition

A random variable X *follows a Poisson distribution of parameter* $\lambda > 0$ *if it is defined on* $\mathbb{N}$ *and*

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k \in \mathbb{N}$$

This distribution depends on only one real positive parameter λ, *we write* $X \sim \mathcal{P}(\lambda)$.

Remark

$$e^x = \sum_{i=0}^{+\infty} \frac{x^i}{i!}$$

So

$$\sum_{k=0}^{\infty} P(X = k) = \sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1$$

Moments of Poisson distribution

If $X \sim \mathcal{P}(\lambda)$ So $E(X) = \lambda$ and $V(X) = \lambda$

Expected value

$$\begin{aligned} E(X) &= \sum_{k=0}^{\infty} kP(X=k) \\ &= \dots \\ &= \lambda. \end{aligned}$$

Variance

- First we calculate $E(X^2) = \sum_{k=0}^{\infty} k^2 P(X=k) = \dots = \lambda(\lambda+1)$.

Then

$$V(X) = \lambda(\lambda+1) - \lambda^2 = \lambda$$

Poisson distribution $\mathcal{P}(\lambda)$

Example

- $X =$ number of laptops sold by day in a shop.
- Suppose that $X \sim \mathcal{P}(5)$.
- The probability of solding 5 laptops by day is

$$P(X = 5) = e^{-5}\frac{5^5}{5!} = e^{-5} \simeq 0.1755$$

- The probability of solding at least 2 laptops is

$$P(X \geqslant 2) = 1 - \left(e^{-5}\frac{5^0}{0!} + e^{-5}\frac{5^1}{1!}\right) \simeq 0.9596$$

- The aveage number of laptops sold by day is 5 since $E(X) = \lambda = 5$.

Properties

If X and Y are two **indenpendent** random variable of Poisson distribution, $X \sim \mathcal{P}(\lambda)$ and $Y \sim \mathcal{P}(\mu)$, Then their sum is also Poisson: $X + Y \sim \mathcal{P}(\lambda + \mu)$.

Approximation of Binomial distribution

$$\text{If } n \to \infty \text{ and } p \to 0 \text{ alors } X : \mathcal{B}(n, p) \sim \mathcal{P}(\lambda)$$

Remark

A good approximationis obtained if $n \geqslant 50$ and $np \leqslant 5$.

In this context, the Poisson distribution is often used to model the number of successes when an experiment with a very low chance of success is repeated a very large number of times.

Applications of Poisson distribution

- The number of persons over 100 years in a community.
- The number of fake phone numbers dialed in one day.
- The number of customers entering a given post office in one day.
- The number of α particles emitted by a radioactive material during a certain period of time.

The variables in these examples are approximately Poisson.

Geometric or Pascal distribution $\mathcal{G}(p)$

Geometric distribution $\mathcal{G}(p)$

- ε : "Repeat a Bernoulli event until the first success".
- Example:

$$\begin{array}{ccccccccc} \bar{A} & \bar{A} & \bar{A} & \bar{A} & \bar{A} & \dots & \bar{A} & \bar{A} & A \\ E & E & E & E & E & \dots & E & E & S \end{array}$$

- Each trial has p as probability of success and $1-p$ as probability of fail.
- $X =$"number of events".

$$\underbrace{E \quad E \quad E \quad E \quad E \quad \dots \quad E \quad E}_{k-1} \quad S$$

- $X(\Omega) = \mathbb{N}^* = \{1, 2, 3, \dots\}$. We say $X \sim \mathcal{G}(p)$.
- $\forall\, k \in \mathbb{N}^* \quad P(X = k) = (1-p)^{k-1}p$
- Attention: Sometimes $X =$ "number of events until having the first success". In this case $X(\Omega) = \mathbb{N}$. We say $X \sim \mathcal{G}(p)$ on $\mathbb{N}$.
- This distribution is often used to model lifetimes, or waiting time, when the time is measured in discrete way (number of days for example).
- Série entière : $\sum_{k=0}^{\infty} x^k = 1/(1-x)$ pour $|x| < 1$
- $\sum_{k=1}^{\infty} P(X = k) = \sum_{k=1}^{\infty}(1-p)^{k-1}p = p\sum_{j=0}^{\infty}(1-p)^j \sum_{k=1}^{\infty}(1-p)^{k-1}p = p\sum_{j=0}^{\infty}(1-p)^j = p\frac{1}{1-(1-p)} = 1$

Moments of Geometric distribution

Expected value

- $E(X) = \sum_{k=1}^{\infty} kP(X = k) = \sum_{k=1}^{\infty} kp(1-p)^{k-1} = p\sum_{k=1}^{\infty} k(1-p)^{k-1}$
- Power series: $\sum_{k=0}^{\infty} x^k = 1/(1-x) \quad \text{pour} \quad |x| < 1$
- 1st derivative: $\sum_{k=1}^{\infty} kx^{k-1} = 1/(1-x)^2$
- So $E(X) = \frac{p}{[1-(1-p)]^2} = \frac{1}{p}$

In other words, if independent trials with probability p of success are performed until the first success occurs, the expected number of trials needed is equal to $1/p$. For example, the expected number of throws of a balanced die that it takes to get the value 1 is 6.

Moments of Geometric distribution

Variance of Geometric distribution

- $V(X) = E(X^2) - E^2(X) = E[X(X-1)] + E(X) - E^2(X)$. While,

$$\begin{aligned} E[X(X-1)] &= \sum_{k=2}^{\infty} k(k-1)p(1-p)^{k-1} \\ &= p(1-p)\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2} \end{aligned}$$

- 1st derivative of Power series: $\sum_{k=1}^{\infty} kx^{k-1} = 1/(1-x)^2$
- 2nd derivative of Power series: $\sum_{k=2}^{\infty} k(k-1)x^{k-2} = 2/(1-x)^3$
- So $E[X(X-1)] = \frac{2p(1-p)}{[1-(1-p)]^3} = \frac{2(1-p)}{p^2}$
- Then $V(X) = E[X(X-1)] + E(X) - E^2(X) = \frac{1-p}{p^2}$.

Negative Binomial distribution

$\mathcal{BN}(r, p)$

Negative Binomial distribution $\mathcal{BN}(r,p)$

- ε : “Repeat a Bernoulli event until r successes”.
- Example with $r = 3$:

$$\begin{array}{ccccccccc} \bar{A} & A & \bar{A} & \bar{A} & \bar{A} & A & \bar{A} & \bar{A} & A \\ E & S & E & E & E & S & E & E & S \end{array}$$

- But we can obtain r successes in other ways:

$$S \quad E \quad E \quad E \quad E \quad E \quad S \quad E \quad S$$

$$E \quad E \quad E \quad E \quad S \quad E \quad S \quad E \quad S$$

- Each trial has p as probability of success and $1 - p$ as probability of fail.
- Let $X =$“number of trails to obtain this result”.

$$\underbrace{\overbrace{E \quad S \quad E \quad E \quad E \quad S \quad E \quad E}^{r-1\,\text{succès and}\,k-r\,\text{échecs}} \quad S}_{X=k}$$

- $X(\Omega) = \{r, r+1, r+2, \dots\}$. We say $X \sim \mathcal{BN}(r,p)$.
- $\forall\, k \in X(\Omega)$,

$$P(X = k) = \binom{k-1}{r-1} p^{r} (1-p)^{k-r}$$

$\mathcal{G}(p) = \mathcal{BN}(1, p)$

- ε : “Repeat a Bernoulli event until r successes”.
- Let,

$$E \quad \ldots \quad E \quad S \quad E \quad \ldots \quad E \quad S \ldots \quad E \ldots \quad E \quad S$$

- Let, Y_1 the number of trials until the first success, Y_2 the number of supplementary trials until the 2nd success, Y_3 until the 3rd success and so on.
- Which means,

$$\underbrace{E \quad \ldots \quad E \quad S}_{Y_1} \quad \underbrace{E \quad \ldots \quad E \quad S}_{Y_2} \quad \underbrace{\ldots}_{\ldots} \quad \underbrace{E \quad \ldots \quad E \quad S}_{Y_r}$$

- The draws being indenpendent and always having the same probability of success, each of the variables $Y_1, Y_2, \ldots, Y_r$ is Geometric $\mathcal{G}(p)$.
- $X =$“number of trials until r successes”$= Y_1 + Y_2 + \ldots + Y_r$.
- So,

$$E(X) = E(Y_1) + E(Y_2) + \ldots + E(Y_r) = \sum_{i=1}^{r} \frac{1}{p} = \frac{r}{p}$$

and

$$V(X) = \sum_{i=1}^{r} V(Y_i) = \frac{r(1-p)}{p^2}$$

since Y_i are indenpendent.