Probabilities

Common Continuous Distributions

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Plan

- X is a **continuous random variable** if it has a non negative **density function** f s.t.

$$P(X \in B) = \int_B f(x)dx$$

- $\int_{-\infty}^{+\infty} f(x)dx = 1$
- $P(a \leqslant X \leqslant b) = \int_a^b f(x)dx$

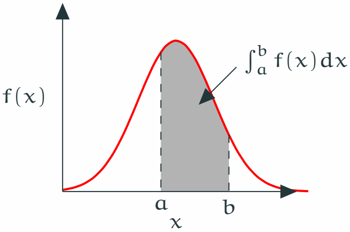


Figure 1: $P(a \leqslant X \leqslant b) =$ *Grey surface*

- Definition: $\forall\ a \in \mathbb{R} \quad F_X(a) = P(X \leqslant a) = \int_{-\infty}^{a} f(x)dx$
- f is the derivative of F
- $F'_X(x) = \frac{d}{dx}F_X(x) = f(x)$
- $P(a < X < b) = P(a \leqslant X \leqslant b) = F_X(b) - F_X(a) = \int_a^b f(x)dx$

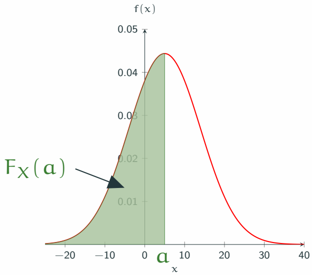


Figure 2: $F_X(a) = P(X < a) =$ *The area of green surface*

Expected value:

- $E(X) = \int_{-\infty}^{+\infty} x f(x) dx$
- $E(aX + b) = aE(X) + b \quad a, b \in \mathbb{R}$

Transfer theorem:

- $E[g(X)] = \int_{-\infty}^{+\infty} g(x) f(x) dx$
- e.g. $E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x) dx$

Variance:

- $V(X) = E\big([X - E(X)]^2\big) = E(X^2) - [E(X)]^2$
- $V(X) \geqslant 0$
- $\forall \quad (a, b) \in \mathbb{R}, V(aX + b) = a^2 V(X)$

Uniform ditribution $U(a, b)$

Definition

The random variable X follows a Uniform distribution on the segment $[a, b]$ with $a < b$ if its density function is given by

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if} \quad x \in [a, b] \\ 0 & \text{if} \quad x \notin [a, b] \end{cases} = \frac{1}{b-a} \mathbb{1}_{[a,b]}(x)$$

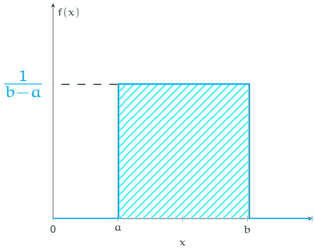


Figure 3: Density function of $\mathcal{U}([a, b])$

Uniform distribution $\mathcal{U}(a, b)$

- The *distribution function* associated to the continuous uniform distribution is

$$F_X(x) = \begin{cases} 0 & \text{if} \quad x < a \\ \dfrac{x-a}{b-a} & \text{if} \quad a \leqslant x \leqslant b \\ 1 & \text{if} \quad x > b \end{cases}$$

- $E(X) = \dfrac{b+a}{2}$
- $V(X) = \dfrac{(b-a)^2}{12}$

Exponential distribution $\mathcal{E}(\lambda)$

Definition

A random variable X is **exponential** (or follows an Exponential distribution) of parameter λ if its density function is given by

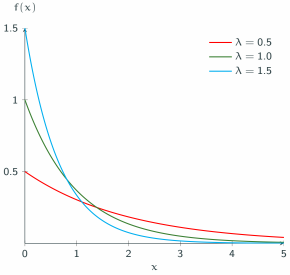
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if} \quad x \geqslant 0 \\ 0 & \text{if} \quad x < 0 \end{cases} = \lambda e^{-\lambda x} \mathbb{1}_{\mathbb{R}^+}(x)$$

We say $X \sim \mathcal{E}(\lambda)$

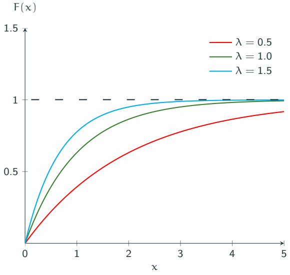
- The distribution function F of an exponential random variable is given by

$$\text{if } x \geqslant 0 \quad F(x) = P(X \leqslant x) = 1 - e^{-\lambda x}$$

- $E(X) = \dfrac{1}{\lambda}$ and $V(X) = \dfrac{1}{\lambda^2}$



(a) Density function of exponential distribution



(b) Distribution function of exponential distribution

Exponential distribution $\mathcal{E}(\lambda)$

Use cases of the exponential distribution:

- Represent the waiting time before the arrival of a specified event.
- To model the lifetime of a phenomenon without memory or without aging.
- A nonnegative random variable X is said to be *memoryless* when

$$P(X > t + h | X > t) = P(X > h) \qquad \forall \quad t, h \geqslant 0$$

- For example, the lifetime of radioactivity or of an electronic component.

Normal distribution or Gaussian

$\mathcal{N}(\mu, \sigma^2)$

Normal distribution $\mathcal{N}(\mu, \sigma^2)$

Definition

A random variable X is said to be **normal** (or Gaussian) with parameters μ and σ^2 if its density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \qquad \forall\, x \in \mathbb{R}$$

With $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$. We say that $X \sim \mathcal{N}(\mu, \sigma^2)$.

Moments of Normal distribution $\mathcal{N}(\mu, \sigma^2)$

- $E(X) = \mu$
- $V(X) = \sigma^2$

Density of Normal distribution $\mathcal{N}(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \qquad \forall\, x \in \mathbb{R}$$

- The function f is **even** with axis of symmetry $x = \mu$ car $f(x+\mu) = f(\mu - x)$.
- $f'(x) = 0$ when $x = \mu$, $f'(x) < 0$ when $x > \mu$ and $f'(x) > 0$ when $x < \mu$

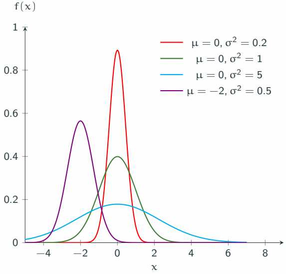


*Figure 5: **Remark:** The parameter μ represents the axis of symmetry and σ the degree of flatness of the curve of the Normal distribution whose shape is that of a bell curve.*

Theorem

- $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$
- $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$
- X_1 et X_2 are independent.

So $X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Standard Normal distribution $\mathcal{N}(0, 1)$

Standard Normal distribution $\mathcal{N}(0,1)$

Definition

A continuous random variable X follows a Standard Normal distribution if its density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \qquad \forall\, x \in \mathbb{R}$$

On dit $X \sim \mathcal{N}(0,1)$.

Moments of Standard Normal distribution$\mathcal{N}(0,1)$

- $E(X) = 0$
- $V(X) = 1$

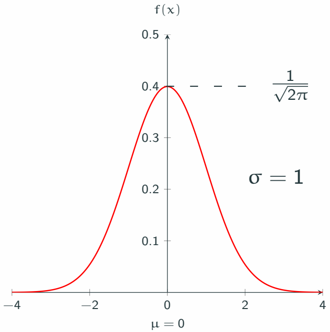


Figure 6: Density of Standard Normal distribution $\mathcal{N}(0,1)$.

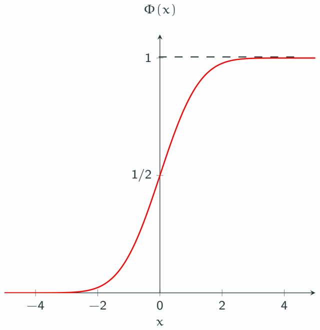


Figure 7: Distribution function of Standard Normal distribution $\mathcal{N}(0,1)$.

Relation between Normal distribution and Standard Normal distribution

Relation between Normal distribution and Standard Normal distribution

Theorem

If X follows a Normal distribution $\mathcal{N}(\mu, \sigma^2)$, then $Z = \dfrac{X - \mu}{\sigma}$ is a random variable that follows the Standard Normal distribution $\mathcal{N}(0, 1)$.

Distribution function of Standard Normal distribution $\mathcal{N}(0, 1)$

Distribution function of Standard Normal distribution $\mathcal{N}(0, 1)$

The distribution function of Standard Normal distribution allows to obtain the probabilities associated to all normal random variables $\mathcal{N}(\mu, \sigma^2)$ after transformation into a standardised variable.

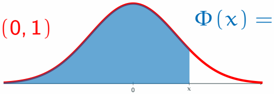
Definition

Let us call Φ, the distribution function of Standard Normal distribution $\mathcal{N}(0, 1)$, such that

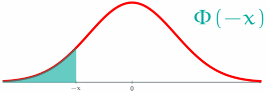
$$\forall\, x \in \mathbb{R} \quad \Phi(x) = P(X \leqslant x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} f(t)dt$$

Distribution function of Standard Normal distribution $\mathcal{N}(0,1)$

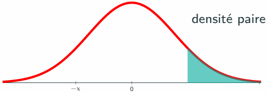
$X \sim \mathcal{N}(0,1)$



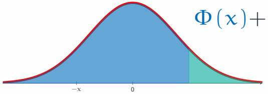
$\Phi(x) = P(X \leqslant x)$



$\Phi(-x) = P(X \leqslant -x)$



densité paire $\Rightarrow$ courbe symétrique % $x = 0$



$\Phi(x) + \Phi(-x) = 1$

Distribution function of Standard Normal distribution $\mathcal{N}(0,1)$

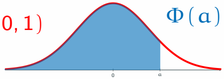
Properties de Φ

The properties associated with the distribution function Φ are:

1. Φ is increasing, continuous and derivable on $\mathbb{R}$ and verifies:
   $\lim_{x \to -\infty} \Phi(x) = 0$ and $\lim_{x \to \infty} \Phi(x) = 1$
2. $\forall\, x \in \mathbb{R} \quad \Phi(x) + \Phi(-x) = 1$
3. $\forall\, x \in \mathbb{R} \quad \Phi(x) - \Phi(-x) = 2\Phi(x) - 1$

Standard Normal distribution table $\mathcal{N}(0,1)$

$X \sim \mathcal{N}(0,1)$

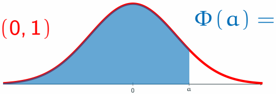


$\Phi(a) = P(X \leqslant a)$

a	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Standard Normal distribution table $\mathcal{N}(0, 1)$

$X \sim \mathcal{N}(0, 1)$



$\Phi(a) = P(X \leqslant a)$

a	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
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1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

For example, for $x = 1.23$ (intersection of the line 1.2 and the column 0.03), we get: $\Phi(1.23) \approx$ 0.8907.

Exemple 1

Let X be a random variable with standard normal distribution. Calculate:

1. $P(X > 2)$
2. $P(2 < X < 5)$

Exemple 2

SLet X be a random variable with normal distribution of parameters $\mu = 3$ et $\sigma^2 = 4$. Calculate:

1. $P(X > 0)$
2. $P(2 < X < 5)$
3. $P(|X - 3| > 4)$

Normal Approximation to the Binomial Distribution

Normal Approximation to the Binomial Distribution

Moivre Laplace Theorem

Suppose that for all n, X_n follows a binomial distribution $\mathcal{B}(n, p)$ with $p \in]0, 1[$.

Then the variable $Z_n = \dfrac{X_n - np}{\sqrt{np(1-p)}}$ converges in law to a Standard Normal distribution $\mathcal{N}(0, 1)$.

- This result was progressively generalized by Laplace, Gauss and others to become the Theorem currently known as the Central Limit Theorem which is one of the two most important results in probability theory.
- In practice, many random phenomena follow approximately a normal distribution.

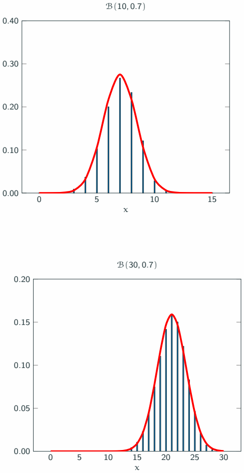

Figure 8: The probability distribution of a random variable $\mathcal{B}(n, p)$ becomes more and more "normal" as n increases.

Other related distributions

Chi-squared distribution χ^2

Definition

Let $X_1, X_2, \dots, X_n$ **standard normal** random variables, and Y the random variable defined by

$$Y = X_1^2 + X_2^2 + \dots + X_i^2 + \dots + X_n^2 = \sum_{i=1}^{n} X_i^2$$

We say that Y follows the χ^2 distribution (or Pearson's law) with n degrees of freedom, $Y \sim \chi^2(n)$

- The χ^2 distribution has many applications in the context of comparison of proportions, tests of conformity of an observed distribution to a theoretical distribution and the test of independence of two qualitative variables. These are the chi-square tests.
- Note: If $n = 1$, the χ^2 corresponds to the square of a standard normal variable $\mathcal{N}(0, 1)$.

Student distribution $St(n)$

Definition

Let U be a random variable following a **standard normal distribution** $\mathcal{N}(0,1)$ and V a random variable following a $\chi^2(n)$ distribution, U and V being independent, we say that $T_n = \dfrac{U}{\sqrt{\frac{V}{n}}}$ follows a Student's distribution with n degrees of freedom. $T_n \sim St(n)$

- Student distribution is used in tests of comparison of parameters (such as the mean) and in the estimation of population parameters from sample data (Student test).

Definition

Let U and V be two independent random variables following a χ^2 distribution with n and m degrees of freedom respectively.

We say that $F = \dfrac{U/n}{V/m}$ follows a Fisher-Snedecor distribution with (n, m) degrees of freedom. $F \sim \mathcal{F}(n, m)$

- The Fisher-Snedecor distribution is used to compare two observed variances and is used especially in the numerous analysis of variance and covariance tests.