Probabilities

Continuous Random Variables

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Plan

Recall: **Discrete** Random Variable

Continuous Random Variable

Distribution function of continuous random variables

Function of a continuous random variable

Moments of Continuous Random Variable

Recall: Discrete Random Variable

Discrete Random Variable

- X is a ***discrete*** random variable if the set of possible values of X, $X(\Omega)$, is finite or countable.
  - The *probability distribution* defined on $X(\Omega)$ by $p_i = p(x_i) = P(X = x_i)$
  - $p(x_i) \geqslant 0$, $\sum_{i=1}^{\infty} p(x_i) = 1$, and $P(a < X \leqslant b) = \sum_{i/a<x_i\leqslant b} p(x_i)$.

Distribution function of a d.r.v.

- The distribution function of X, that we note $F_X(a)$, defined for each real number a, $-\infty < a < \infty$, by $F_X(a) = P(X \leqslant a) = \sum_{i/x_i \leqslant a} P(X = x_i)$.
  - Staircase function.
  - $F_X(a) \leqslant 1$ (it is a probability).
  - $F_X(a)$ is continuous at right.
  - $\lim\limits_{a \to -\infty} F_X(a) = 0$ et $\lim\limits_{a \to \infty} F_X(a) = 1$
  - $P(a < X \leqslant b) = F(b) - F(a) \qquad$ pour tout $a < b$

Moments of d.r.v.

- $E(X) = \sum_{i \in \mathbb{N}} x_i p(x_i)$
- $V(X) = E(X^2) - E^2(X)$

Continuous Random Variable

- Previously we have dealt with Discrete Random Variables, i.e. variables whose universe is finite or countable.
- There are however variables whose universe is infinite uncountable.
- Examples:
  - The arrival time of a train at a given station.
  - The lifetime of a transistor.

Definition

X is a continuous random variable [1] ***with density*** *if there exists a non-negative function f defined for any $x \in \mathbb{R}$ and verifying for any set B of real numbers the property*

$$P(X \in B) = \int_B f(x)dx$$

The function f is called density function of the random variable X.

- All probability questions related to X can be treated with f.
- For example if $B = [a, b]$, we get:

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)dx$$

[1] Not all Continuous Random Variable have a density function.

Graphically, $P(a \leqslant X \leqslant b)$ is the area of the surface between the x-axis, the curve corresponding to $f(x)$ and the lines $x = a$ and $x = b$.

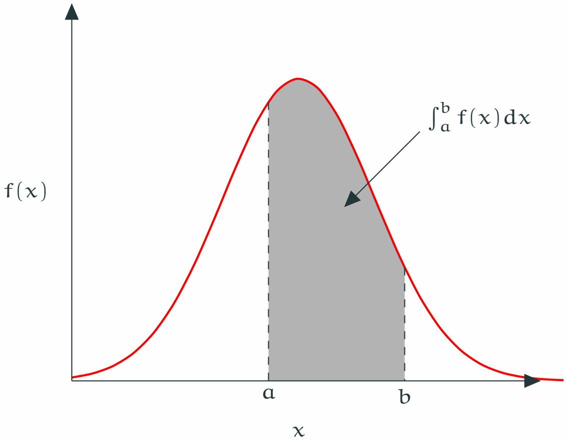


Figure 1: $P(a \leqslant X \leqslant b) =$ *area of shaded surface*

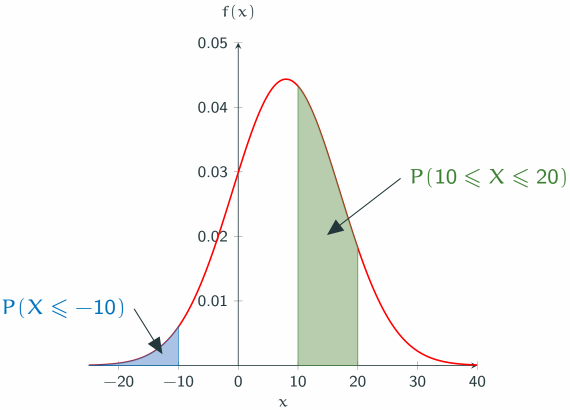
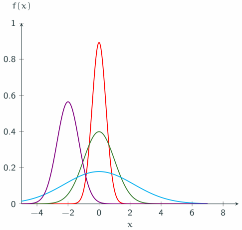
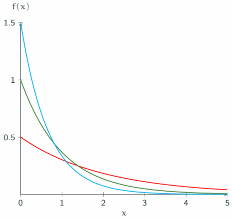
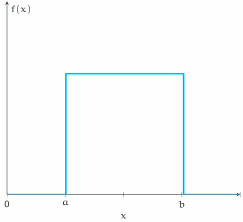


Figure 2: The colored areas corresponds to probabilities. $f(x)$ *being a probability density function.*







Proprieties

For any continuous random variable X of density f:

- $f(x) \geqslant 0 \quad \forall x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x)dx = 1$
- Since $P(a \leqslant X \leqslant b) = \int_a^b f(x)dx$, if $a = b$ then $P(X = a) = \int_a^a f(x)dx = 0$
- This means that the probability of a continuous random variable taking a fixed isolated value is always zero.

Continuous Random Variable - Examples

Example

Let X be the random real variable of probability density

$$f(x) = \begin{cases} kx & \text{if} \quad 0 \leqslant x \leqslant 5 \\ 0 & \text{if not} \end{cases}$$

1. Calculate k.
2. Calculate: $P(1 \leqslant X \leqslant 3), P(2 \leqslant X \leqslant 4)$ and $P(X < 3)$.

Example

Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{1}{6}x + k & \text{if} \quad 0 \leqslant x \leqslant 3 \\ 0 & \text{if not} \end{cases}$$

1. Calculate k.
2. Calculate $P(1 \leqslant X \leqslant 2)$

Distribution function of continuous random variables

Definition

If as for Random Variable Discrete, we define the distribution function of X by:

$$\begin{aligned} F_X : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto F_X(a) = P(X \leqslant a) \end{aligned}$$

then the relation between the distribution function F_X and the probability density function $f(x)$ is the following:

$$\forall \quad a \in \mathbb{R} \quad F_X(a) = P(X \leqslant a) = \int_{-\infty}^{a} f(x)dx$$

Proprieties

For a continuous random variable X:

- $F'_X(x) = \frac{d}{dx} F_X(x) = f(x)$.
- For all real numbers $a \leqslant b$,

$$\begin{aligned} P(a < X < b) &= P(a < X \leqslant b) = P(a \leqslant X < b) \\ &= P(a \leqslant X \leqslant b) = F_X(b) - F_X(a) = \int_a^b f(x)dx \end{aligned}$$

The distribution function corresponds to the cumulative probabilities associated with the continuous random variable on an interval.

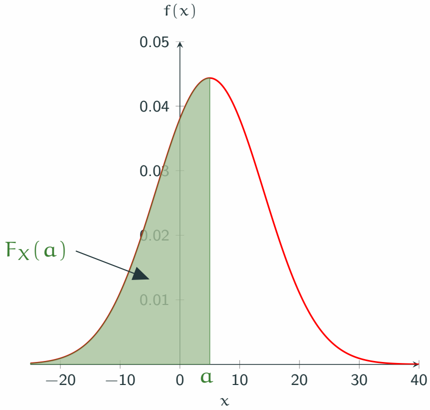


Figure 3: The area shaded in green under the curve of the density function corresponds to the probability $P(X < a) = F_X(a)$ *and is* 0.5 *because this corresponds exactly to half of the total area under the curve.*

Proprieties

The properties of the distribution function are as follows:

1. F_X is continuous on $\mathbb{R}$, derivable at any point where f is continuous.
2. F_X is increasing on $\mathbb{R}$.
3. F_X has values in $[0, 1]$.
4. $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to +\infty} F_X(x) = 1$.

Example

Let X and Y two random variables of density functions:

$$f_X(x) = \begin{cases} kx & \text{if} \quad 0 \leqslant x \leqslant 5 \\ 0 & \text{if not} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \frac{1}{6}y + k & \text{if} \quad 0 \leqslant y \leqslant 3 \\ 0 & \text{if not} \end{cases}$$

Calculate $F_X(a)$ and $F_Y(a)$ for all $a \in \mathbb{R}$.

Function of a continuous random variable

- Let X be a continuous random variable with density f_X and distribution function F_X.
- Let h be a continuous function defined on $X(\Omega)$, then $Y = h(X)$ is a random variable.
- To determine the density of Y, denoted f_Y, we first compute the distribution function of Y, denoted F_Y, then we derivate it to determine f_Y.

Calculating the densities

Let X be a continuous random variable with density f_X and distribution function F_X. Find the density function of the following random variables:

- $Y = aX + b$
- $Z = X^2$
- $T = e^X$

Example

Let X a random variable having the density function:

$$f_X(x) = 2x \times \mathbb{1}_{[0,1]}(x)$$

Determine the density function of: $Y = 3X + 1$, $Z = X^2$ and $T = e^X$.

Moments of Continuous Random Variable

Definition

If X is a continuous random variable of density f, we call the expected value of X, the real $E(X)$, defined by:

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx$$

if it exists.

The properties of the expected value of a continuous random variable are the same as for a discrete random variable.

Proprieties

Let X be a continuous random variable,

- $E(aX + b) = aE(X) + b \qquad a \geqslant 0$ and $b \in \mathbb{R}$.
- If $X \geqslant 0$ then $E(X) \geqslant 0$.
- If X and Y are two Random Variables defined on the same universe Ω then

$$E(X + Y) = E(X) + E(Y)$$

Theorem

If X is a random variable of density $f(x)$, then for any real function g we have

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)dx$$

Example

Let X a random variable of density

$$f_X(x) = \begin{cases} 2x & \text{if} \quad 0 \leqslant x \leqslant 1 \\ 0 & \text{if not} \end{cases}$$

Calculate the expected value of $Y = 3X + 1$, $Z = X^2$ and $T = e^X$.

Variance of a continuous random variable

The variance of a random variable $V(X)$ is a dispersion parameter which corresponds to the centered moment of order 2 of the random variable X.

Definition

If X is a random variable with expectation $E(X)$, we call the variance of X the real

$$V(X) = E\left([X - E(X)]^2\right) = E(X^2) - [E(X)]^2$$

If X is a continuous random variable, we compute $E(X^2)$ using the transfer theorem,

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\,dx$$

Example

Calculate la variance of X defined in the previous example.

Proprieties

If X is a random variable with a variance then:

- $V(X) \geqslant 0$, if it exists.
- $\forall\ a \in \mathbb{R}, V(aX) = a^2 V(X)$
- $\forall\ (a, b) \in \mathbb{R}, V(aX + b) = a^2 V(X)$
- If X and Y are two independent Random Variables, $V(X + Y) = V(X) + V(Y)$

Definition

If X is a random variable with variance $V(X)$, we call the standard deviation of X the real:

$$\sigma_X = \sqrt{V(X)}$$