Probabilities

Continuous Random Variables

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Plan

Recall: **Discrete** Random Variable

Continuous Random Variable

Distribution function of continuous random variables

Function of a continuous random variable

Moments of Continuous Random Variable

$\sum$

$\int$

Recall: Discrete Random Variable

Discrete Random Variable

$X(\Omega)$ takes a finite values

- X is a ***discrete*** random variable if the set of possible values of X, $X(\Omega)$, is finite or countable.
  - The *probability distribution* defined on $X(\Omega)$ by $p_i = p(x_i) = P(X = x_i)$
  - $p(x_i) \geqslant 0$, $\sum_{i=1}^{\infty} p(x_i) = 1$, and $P(a < X \leqslant b) = \sum_{i/a<x_i \leqslant b} p(x_i)$.

$\mathbb{N}$

$X=1$

$X=2$

Recall: Discrete Random Variable

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Distribution function of a d.r.v.

- The distribution function of X, that we note $F_X(a)$, defined for each real number a, $-\infty < a < \infty$, by $\boxed{F_X(a) = P(X \leqslant a)} = \sum_{i/x_i \leqslant a} P(X = x_i)$.

$\forall a \in \mathbb{R}$

$F_X(a) = P(X < a)$

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  - Staircase function.
  - $F_X(a) \leqslant 1$ (it is a probability).
  - $F_X(a)$ is continuous at right.
  - $\lim\limits_{a\to-\infty} F_X(a) = 0$ et $\lim\limits_{a\to\infty} F_X(a) = 1$
  - $P(a < X \leqslant b) = F(b) - F(a)$ pour tout $a < b$

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Moments of d.r.v.

$\sum x_i^2 p(x_i) \leftarrow$ transfer theorem

- $E(X) = \sum_{i\in\mathbb{N}} x_i p(x_i)$
- $V(X) = E(X^2) - E^2(X)$ $= E([X - E(X)]^2)$

$E(g(x)) = \sum g(x_i) p(x_i)$

Continuous Random Variable

Continuous Random Variable - Density function and Probabilities

- Previously we have dealt with Discrete Random Variables, i.e. variables whose universe is finite or countable.
- There are however variables whose universe is infinite uncountable.
- Examples:
  - The arrival time of a train at a given station.
  - The lifetime of a transistor.

unlimited nb of possibilities

$X(\Omega) = \mathbb{R}$

$\boxed{[0,1]}$

$\cancel{\{0,1\}}$

$X(\Omega) = [0,1]$

$$f(x) = x \times \mathbb{1}_{[0,1]}(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 0 & \text{if not} \end{cases}$$

[1] Not all Continuous Random Variable have a density function.

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- Examples:
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Definition

*X is a continuous random variable [1] **with density** if there exists a non-negative function f defined for any $x \in \mathbb{R}$ and verifying for any set B of real numbers the property*

$B \subset \mathbb{R}$

$$\boxed{P(X \in B) = \int_B f(x)\,dx}$$

two conditions:

* $f \geq 0$
* $\int_{\mathbb{R}} f(x)\,dx = 1$

$P(X(\Omega))$

The function f is called density function of the random variable X.

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Definition

X is a continuous random variable [1] ***with density*** *if there exists a non-negative function f defined for any $x \in \mathbb{R}$ and verifying for any set B of real numbers the property*

$$P(X \in B) = \int_B f(x)dx$$

The function f is called density function of the random variable X.

- All probability questions related to X can be treated with f.
- For example if $B = [a, b]$, we get:

$$P(X \in B) = P(a \leqslant X \leqslant b) = \int_a^b f(x)dx$$

$P(X \in [1,2])$

$= P(1 \leq X \leq 2)$

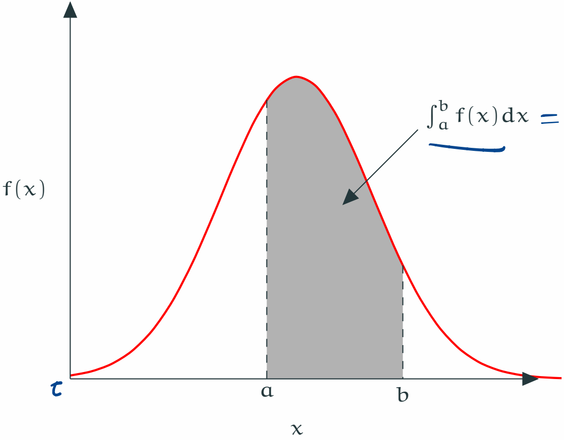
$= \int_1^2 f(x)dx$

[1] Not all Continuous Random Variable have a density function.

Graphically, $P(a \leqslant X \leqslant b)$ is the area of the surface between the x-axis, the curve corresponding to $f(x)$ and the lines $x = a$ and $x = b$.

$\int_{\mathbb{R}} f(x)\,dx = 1$

total area of the surface between f and x-axis



$= P(a \leq X \leq b)$

Figure 1: $P(a \leqslant X \leqslant b) =$ *area of shaded surface*

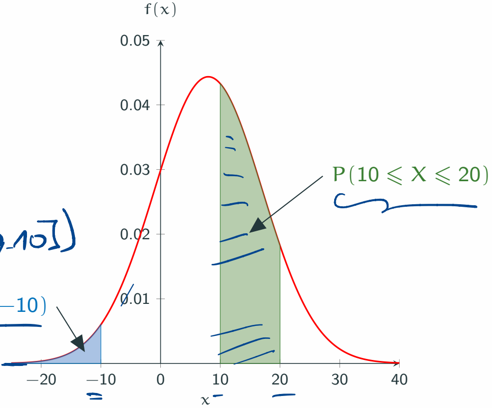


Figure 2: The colored areas corresponds to probabilities. $f(x)$ *being a probability density function.*

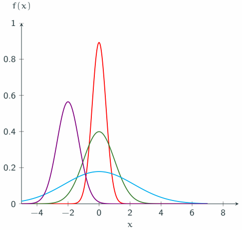
$X(\Omega) = \mathbb{R}^+$

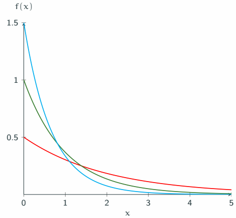
$P(X \leq 2)$

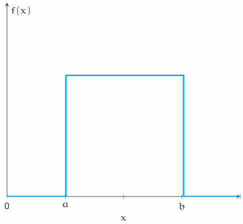
$\int_0^2 f(x)dx$

$P(X \in]-\infty, 10])$

$\int_{-\infty}^2 f(x)dx = \underbrace{\int_{-\infty}^0 f(x)dx}_{=0} + \int_0^2 f(x)dx$







Proprieties

For any continuous random variable X of density f:

$P(X=a)=0$

- $f(x) \geqslant 0 \quad \forall x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x)dx = 1$
- Since $P(a \leqslant X \leqslant b) = \int_a^b f(x)dx$, if $a = b$ then $P(X = a) = \int_a^a f(x)dx = 0$
- This means that the probability of a continuous random variable taking a fixed isolated value is always zero.

$\int_{\mathbb{R}} f(x)\,dx = \int_{\infty}^{+\infty} f(x)\,dx = 1$

$P(X=a) = P(a \leq X \leq a) = \int_a^a f(x)dx = 0$

Example

Let X be the random real variable of probability density

$$f(x) = \begin{cases} \overset{\text{constant}}{k}x & \text{if } 0 \leqslant x \leqslant 5 \\ 0 & \text{if not} \end{cases}$$

$X(\Omega) = [0,5]$.

1. Calculate k.
2. Calculate: $P(1 \leqslant X \leqslant 3), P(2 \leqslant X \leqslant 4)$ and $P(X < 3)$.

1) we know that $\int_{\mathbb{R}} f(x)dx = 1$, $\int_{\mathbb{R}} f(x)dx = \int_0^5 kx\,dx = \left[k\frac{x^2}{2}\right]_0^5 = k\left(\frac{25}{2} - \frac{0}{2}\right) = \frac{25k}{2}$

so $k = 2/25$.

2) * $P(1 \leq X \leq 3) = \int_1^3 f(x)dx = \frac{2}{25}\left[\frac{x^2}{2}\right]_1^3 = \frac{2}{25}\left(\frac{8}{2}\right) = 8/25$.

* $P(2 \leq X \leq 4) = \int_2^4 f(x)dx = ---$

* $P(X < 3) = \int_{-\infty}^3 f(x)dx = \int_0^3 \frac{2}{25}x\,dx = \frac{2}{25}\left[\frac{x^2}{2}\right]_0^3 = \frac{9}{25}$.

Note that $P(X<3) = P(X \leq 3)$

Example

Let X be the random real variable of probability density

$$f(x) = \begin{cases} kx & \text{if } 0 \leqslant x \leqslant 5 \\ 0 & \text{if not} \end{cases}$$

1. Calculate k.
2. Calculate: $P(1 \leqslant X \leqslant 3), P(2 \leqslant X \leqslant 4)$ and $P(X < 3)$.

Example

Let X be a continuous random variable with density function

$$f(x) = \begin{cases} \frac{1}{6}x + k & \text{if } 0 \leqslant x \leqslant 3 \\ 0 & \text{if not} \end{cases}$$

1. Calculate k.
2. Calculate $P(1 \leqslant X \leqslant 2)$

$= \int_1^2 \left(\frac{1}{6}x + \frac{1}{12}\right) dx = \cdots$

$\int_{\mathbb{R}} f(x)dx = \int_0^3 \left(\frac{1}{6}x + k\right)dx = \left[\frac{1}{6}\frac{x^2}{2} + kx\right]_0^3$

$= \frac{3}{4} + 3k = 1$ So $k = 1/12$.

fonction de répartition

Distribution function of continuous random variables

Definition

If as for Random Variable Discrete, we define the distribution function of X *by:*

Since X is continuos

$$\begin{aligned} F_X : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto F_X(a) = P(X \leqslant a) = P(X < a) \end{aligned}$$

then the relation between the distribution function F_X *and the probability density function* $f(x)$ *is the following:*

$$\forall \quad a \in \mathbb{R} \quad F_X(a) = P(X \leqslant a) = \int_{-\infty}^{a} f(x)\,dx$$

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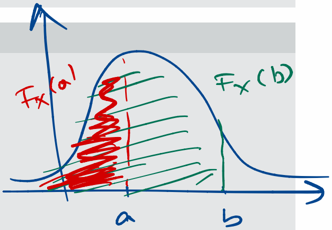
Proprieties

For a continuous random variable X:

- $F'_X(x) = \frac{d}{dx}F_X(x) = f(x).$ when F_X is derivable.
- For all real numbers $a \leqslant b$,

$$P(a < X < b) = P(a < X \leqslant b) = P(a \leqslant X < b)$$

$$= P(a \leqslant X \leqslant b) = F_X(b) - F_X(a) = \int_a^b f(x)dx$$



Distribution function of continuous random variables

The distribution function corresponds to the cumulative probabilities associated with the continuous random variable on an interval.

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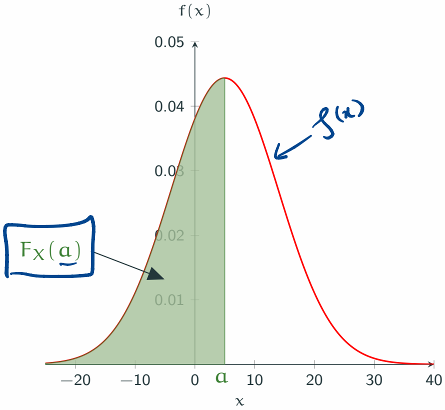
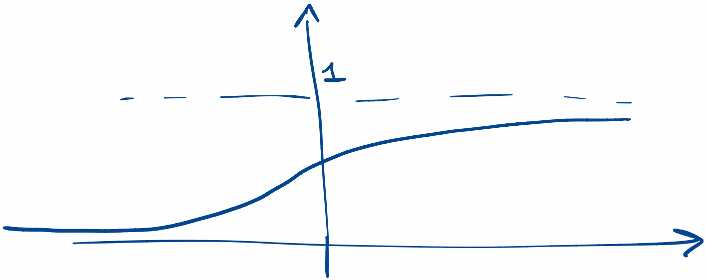


Figure 3: The area shaded in green under the curve of the density function corresponds to the probability $P(X < a) = F_X(a)$ *and is* 0.5 *because this corresponds exactly to half of the total area under the curve.*

Proprieties

The properties of the distribution function are as follows:

1. F_X is continuous on $\mathbb{R}$, derivable at any point where f is continuous.
2. F_X is increasing on $\mathbb{R}$.
3. F_X has values in $[0, 1]$.
4. $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to +\infty} F_X(x) = 1$.



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4. $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to +\infty} F_X(x) = 1$.

$\forall a \in \mathbb{R}, F_X(a) = P(X \leq a)$

* if $a \leq 0$, $F_X(a) = 0$
* if $a \geq 5$, $F_X(a) = 1$
* if $a \in [0, 5]$

$F_X(a) = P(X \leq a)$

$= \int_0^a f(x)dx = \frac{2}{25}\left[\frac{x^2}{2}\right]_0^a$

$= a^2/25$

Example

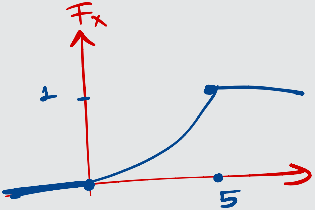
Let X and Y two random variables of density functions:

$$f_X(x) = \begin{cases} \overset{2/25}{k}x & \text{if } 0 \leqslant x \leqslant 5 \\ 0 & \text{if not} \end{cases}$$

and

$$f_Y(y) = \begin{cases} \frac{1}{6}y + k & \text{if } 0 \leqslant y \leqslant 3 \\ 0 & \text{if not} \end{cases}$$

Calculate $\boxed{F_X(a)}$ and $F_Y(a)$ for all $a \in \mathbb{R}$.



X Let $Y = g(X)$

Function of a continuous random variable

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Calculating the densities

Let X be a continuous random variable with density f_X and distribution function F_X. Find the density function of the following random variables:

- $Y = aX + b$
- $Z = X^2$
- $T = e^X$

$* \ \forall z \in Z(\Omega)$

$F_Z(z) = P(Z \leq z)$

$= P(X^2 \leq z)$

$= P(-\sqrt{z} \leq X \leq +\sqrt{z})$

$= \int_{-\sqrt{z}}^{+\sqrt{z}} f_X(x)dx$ or $F_X(\sqrt{z}) - F_X(-\sqrt{z})$

$X^2 \leq z$

$-\sqrt{z} \leq X \leq \sqrt{z}$

$X(\Omega)$

$Z = X^2$

$F_Z \ / f_Z$

So $f_Z(z) = (F_Z(z))' = \frac{1}{2\sqrt{z}} f_X(\sqrt{z}) + \frac{1}{2\sqrt{z}} f_X(-\sqrt{z}) = \frac{1}{2\sqrt{z}} [f_X(\sqrt{z}) + f_X(-\sqrt{z})]$

- Let X be a continuous random variable with density f_X and distribution function F_X.
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$* \; Z(\Omega) = [0,1]$

$\bullet$ if $z \leq 0$, $F_Z(z) = 0$; if $z \geq 1$, $F_Z(z) = 1$

$\bullet$ if $z \in [0,1]$, $F_Z(z) = P(Z \leq z)$

$= P(X^2 \leq z) = P(\cancel{-\sqrt{z}}^{\,0} \leq X \leq \sqrt{z})$ $X(\Omega)$

$= \int_0^{\sqrt{z}} 2x\,dx = [x^2]_0^{\sqrt{z}} = z.$

So $F_Z(z) = \begin{cases} 0 & \text{if } z \leq 0 \\ z & \text{if } 0 \leq z \leq 1 \\ 1 & \text{if } z \geq 1 \end{cases}$

So $f_Z(z) = 1 \times \mathbb{1}_{[0,1]}(z)$.

Example

Let X a random variable having the density function:

$X(\Omega) = [0,1]$

$$f_X(x) = 2x \times \mathbb{1}_{[0,1]}(x)$$

Determine the density function of: $Y = 3X + 1$, $Z = X^2$ and $T = e^X$.

Moments of Continuous Random Variable

$f_X(x) = 2x \times \mathbb{1}_{[0,1]}(x)$

$T = e^X$

$* \ T(\Omega) = [1, e]$

$* \ F_T(t) = \begin{cases} 0 \text{ if } t \leq 1 \\ 1 \text{ if } t \geq e \\ P(T \leq t) \text{ if } t \in [1, e] \end{cases}$

$\mathbb{1}_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \in \bar{A} \end{cases}$

if $t \in [1, e]$,

$F_T(t) = P(T \leq t)$

$= P(e^X \leq t)$

$= P(X \leq \ln(t))$

$= \int_0^{\ln(t)} 2x \, dx = [x^2]_0^{\ln t}$

$F_T(t) = \ln^2 t.$

So $f_T(t) = \frac{dF_T(t)}{dt} = \frac{2 \ln t}{t} \times \mathbb{1}_{[1,e]}(t).$

Definition

If X is a continuous random variable of density f, we call the expected value of X, the real $E(X)$, defined by:

$$E(X) = \int_{-\infty}^{+\infty} x f(x) \, dx$$

if it exists.

Expected value

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Proprieties

Let X be a continuous random variable,

- $E(aX + b) = aE(X) + b \qquad a \geqslant 0$ and $b \in \mathbb{R}$.
- If $X \geqslant 0$ then $E(X) \geqslant 0$.
- If X and Y are two Random Variables defined on the same universe Ω then

$$E(X + Y) = E(X) + E(Y)$$

Theorem

If X is a random variable of density $f(x)$, then for any real function g we have

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) f(x) dx$$

$$E(X^2) = \sum x_i^2 P(X = x_i)$$

$$E(X^2) = \int_{\mathbb{R}} x^2 f(x) dx.$$

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Example

Let X a random variable of density

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant 1 \\ 0 & \text{if not} \end{cases}$$

Calculate the expected value of $Y = 3X + 1$, $Z = X^2$ and $T = e^X$.

$\bullet\ E(X) = \int_{\mathbb{R}} x f(x)\,dx = \int_0^1 x(2x)\,dx = \int_0^1 2x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^1 = 2/3.$

$\bullet\ E(Z) = E(X^2) = \int_{\mathbb{R}} x^2 f(x)\,dx = \int_0^1 x^2(2x)\,dx = 2\left[\frac{x^4}{4}\right]_0^1 = 1/2 \leftarrow$

$\bullet\ E(T) = \int_0^1 e^x f(x)\,dx = \int_0^1 2x e^x\,dx$

$2\int_0^1 x e^x\,dx$

der $\left(\begin{array}{ccc} x & \xrightarrow{+} & e^x \\ 1 & \xrightarrow{-} & e^x \\ 0 & & e^x \end{array} \right)$ int

$= 2\left[[xe^x]_0^1 - [e^x]_0^1 \right]$

$= 2(1 - e + 1) = 2(2 - e).$

Variance of a continuous random variable

The variance of a random variable $V(X)$ is a dispersion parameter which corresponds to the centered moment of order 2 of the random variable X.

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Definition

If X is a random variable with expectation $E(X)$, we call the variance of X the real

$$V(X) = E\left([X - E(X)]^2\right) = E(X^2) - [E(X)]^2$$

If X is a continuous random variable, we compute $E(X^2)$ using the transfer theorem,

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Example

Calculate la variance of X defined in the previous example.

$f_X(x) = 2x \times \mathbb{1}_{[0,1]}(x)$

$E(X) = 2/3$, $E(X^2) = \int_0^1 x^2 f(x)\,dx = \ldots = 1/2$

$V(X) = E(X^2) - E^2(X) = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \_\_ \_\_ .$

Proprieties

If X is a random variable with a variance then:

- $V(X) \geqslant 0$, if it exists.
- $\forall\ a \in \mathbb{R}, V(aX) = a^2V(X)$
- $\forall\ (a, b) \in \mathbb{R}, V(aX + b) = a^2V(X)$
- If X and Y are two independent Random Variables, $V(X + Y) = V(X) + V(Y)$

$$V(X+Y) = V(X) + V(Y) + 2\,\mathrm{cov}(X,Y)$$

Proprieties

If X is a random variable with a variance then:

- $V(X) \geqslant 0$, if it exists.
- $\forall\ a \in \mathbb{R}, V(aX) = a^2V(X)$
- $\forall\ (a,b) \in \mathbb{R}, V(aX+b) = a^2V(X)$
- If X and Y are two independent Random Variables, $V(X+Y) = V(X) + V(Y)$

Definition

If X is a random variable with variance $V(X)$, we call the standard deviation of X the real:

$$\sigma_X = \sqrt{V(X)}$$