Probability

Dicrete Random Variables

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Plan

Introduction to probability theory

Measurable space

Randomness (Uncertainty)

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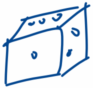
Randomness (Uncertainty)

Fundamental example: consider the game of a die throw.



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Random experience

experiment

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$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

finite

countable

Omega ← set

$\Omega = \{$

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subset: $\{1,2\} \subset \Omega$

element: $\{1,2\} \in \mathcal{P}(\Omega)$

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- The The Power set of Ω, called $\mathcal{P}(\Omega)$, is the set of all subsets of Ω.

$$\mathcal{P}(\Omega) = \{\emptyset, \{1\}, \{2\}, \{3\}, ---, \{1,2\}, \{2,3\}, \_\_\_\_, \{3,4,5\}, \_\_\_\_, \Omega\}$$

$\emptyset$

$\{1\}$

$\{2,3\}$

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Ω

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$\mathcal{A} \subseteq \mathcal{P}(\Omega)$

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- **σ-Algebra**: We call σ-Algebra any family $\mathcal{A}$ of subsets of Ω satisfying:
  1. $\Omega \in \mathcal{A}$.
  2. if $A \in \mathcal{A}$, then $\bar{A} \in \mathcal{A}$.
  3. if $(A_n)_{n\in\mathbb{N}}$ is a sequence of elements in $\mathcal{A}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

$A = \{1,2\} \in \mathcal{P}(\Omega)$

$\bar{A} = \{3,4,5,6\} \in \mathcal{P}(\Omega)$

$B = \{2,4,6\}$

$A \cup B = \{1,2,4,6\} \in \mathcal{P}(\Omega)$

Tribu

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- $(\Omega, \mathcal{A})$ is a measurable space (or a Borel space).

Notions on Events

Event $\Longleftrightarrow$ (Subset of Ω) (element of $\mathcal{A}$)

- Let $(\Omega, \mathcal{A})$ be a measurable space:
  - The set $\mathcal{A}$ is called σ-Algebra of events. The elements of $\mathcal{A}$ are called the events.
  - The event Ω is called certain event. The event $\emptyset$ is called impossible event.

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- Operations on events. Let A and B be two events:
  - $\bar{A}$ is the complement event of A (we also note A^c). barA=Om $\setminus$ A.
    barA occurs if and only if A does not occur.
  - $A \cap B$ is the event «A and B».
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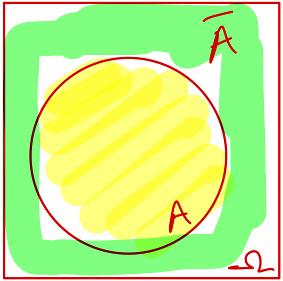

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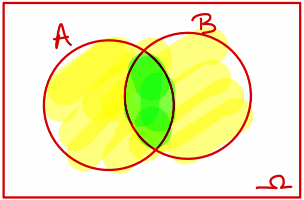

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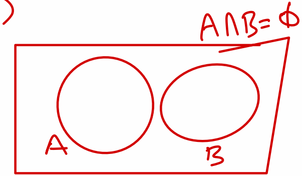
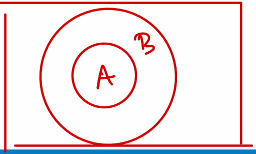
- Mutually Exclusive Events: A and B are mutually exclusive if their simultaneous realization is impossible: $A \cap B = \emptyset$. → incompatible events (disjoint events)



- Implication: A implies B means that if A occurs, then B also occurs: $A \subset B$.

$A \Rightarrow B$



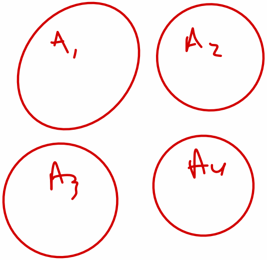
- Let $(\Omega, \mathcal{A})$ a measurable space. A probability function on $(\Omega, \mathcal{A})$, is any map

$$P : \mathcal{A} \to \mathbb{R}$$

such that:

1. $\forall A \in \mathcal{A}, P(A) \geqslant 0$.
2. $P(\Omega) = 1$.
3. $\forall (A_n)_{n \in \mathbb{N}^*} \in \mathcal{A}^{\mathbb{N}^*}$, a familty of pairwise disjoint (mutually exclusive) events, we have:

$$P\Big(\bigcup_{n \in \mathbb{N}^*} A_n\Big) = \sum_{n=1}^{+\infty} P(A_n)$$



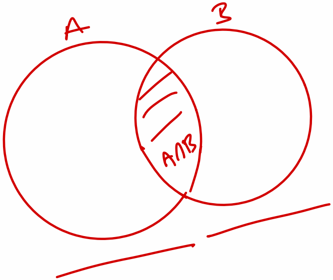
- The triplet $(\Omega, \mathcal{A}, P)$ is called a **probability space**.

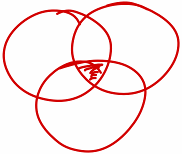
Probability: Properties

1. $P(\emptyset) = 0$.
2. $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$.
3. If A_1 and A_2 are mutually exclusive, $A_1 \cap A_2 = \emptyset$, $P(A_1 \cup A_2) = P(A_1) + P(A_2)$.
4. $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$.
5. $P(\bar{A}) = 1 - P(A)$.
6. $P(B \setminus A) = P(B) - P(B \cap A)$.
7. $A \subset B \Rightarrow P(A) \leqslant P(B)$.

$P(\Omega) = 1$

$P(\bar{A}) = P(\Omega \setminus A) = P(\Omega) - P(A) = 1 - P(A)$





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Uniform probability on finite Ω

- Let Ω be a finite sample space. We say that P is the **uniform probability** on the measurable space $(\Omega, P(\Omega))$ if:

$$\forall \omega, \omega' \in \Omega, \qquad P(\{\omega\}) = P(\{\omega'\})$$

One also says that there is **equiprobability** of elementary events.

- Let $(\Omega, \mathcal{P}(\Omega), P)$ be a finite probability space. If P is the uniform probability, then

$$\forall A \in \mathcal{A}, \qquad P(A) = \frac{\mathrm{Card}(A)}{\mathrm{Card}(\Omega)}$$

Conditional probability

- Let $(\Omega, \mathcal{A}, P)$ be a probability space and $B \in \mathcal{A}$ such that $P(B) > 0$. The map function P_B defined on $\mathcal{A}$ by:

$$P_B(A) = P(A|B) = \frac{P(A \cap B)}{P(B)}, \qquad \forall A \in \mathcal{A}$$

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- Chain rule:

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \qquad P(B/A) = \frac{P(B \cap A)}{P(A)}$$

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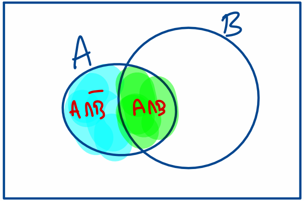

- Law of total probability:
  - $\forall A \in \mathcal{A}, \quad P(A) = P(A \cap B) + P(A \cap \bar{B})$

are mutually exclusive

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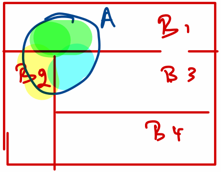

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  - $\forall A \in \mathcal{A}, \quad P(A) = P(A \cap B) + P(A \cap \bar{B})$
  - We call partition of Ω, a set of events that are pairwise disjoint and whose union is the sample space Ω. The partition is said to be "countable" if its cardinality is at most equal to that of $\mathbb{N}$.
  - Let $(B_n)_{n \geqslant 0}$ a partition of Ω. We have:

$$\forall A \in \mathcal{A}, \qquad P(A) = \sum_{n \geqslant 0} P(A \cap B_n)$$

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A+B are indep $\Leftrightarrow$ $P(A/B) = P(A)$

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- Independence: Events A and B are independent iff $P(A \cap B) = P(A)P(B)$. (or $P(B/A) = P(B)$)

First Bayes' theorem

Let $(\Omega, \mathcal{A}, P)$ a probability space. For all events A and B such that $P(A) \neq 0$ and $P(B) \neq 0$, we have:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

Second Bayes' theorem

Let $(\Omega, \mathcal{A}, P)$ a probability space and $(B_n)_{n\geqslant 0}$ a partition of Ω s.t. for all $n \geqslant 0$ $P(B_n) \neq 0$. We have for all $A \in \mathcal{A}$ s.t. $P(A) \neq 0$

chain Rule

$$P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_{n\geqslant 0} P(A|B_n)P(B_n)} \quad \forall i \geqslant 0$$

total prob. law

Break

9:10 → 9:20

Real Random Variable

SwS 01679

Concept of Real Random Variable

Definition

Let ε an experiment and $(\Omega, \mathcal{A}, \mathrm{P})$ the associated probability space. In many situations, one associates to each result $\omega \in \Omega$ a real number denoted $\mathrm{X}(\omega)$; In this way, one builds a map $\mathrm{X} : \Omega \to \mathbb{R}$. Historically, ε was a game and X représented the earning of a player.

$P : \mathcal{A} \longrightarrow \mathbb{R}$ $[0,1]$

σ-Algebra

$X : \Omega \longrightarrow \mathbb{R}$

also a map

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Example: a die throw

A player throws a fair six faces dice and we observe the obtained number:

- If the result is 1,3 or 5, the player earns 1 euro.
- If the result is 2 or 4, the player earns 5 euros.
- If the result is 6, the player loses 10 euros.

Analysis

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$(\Omega, \mathcal{A})$ measurable space

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$(\Omega, \mathcal{A}, \mathbb{P})$ is the probability space

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So we have

- $X(1) = X(3) = X(5) = 1$ (+1 €)
- $X(2) = X(4) = 5$ (+5 €)
- $X(6) = -10$ (−10 €)

We say that X is a **random variable** on Ω.

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$P(X=1) = P(\text{"obtain 1 or 3 or 5"}) = P(\{1,3,5\}) = \frac{|\{1,3,5\}|}{|\Omega|} = 3/6 = 1/2.$

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Similarly, we have:

One can ask what is the probability for the player to win 1 euro:

$\Rightarrow$ $X(\omega) = 1$.

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Concept of Real Random Variable

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Similarly, we have:

- $P(X = 5) = 1/3$.
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In this chapter:

- We treat the case where $X(\Omega)$ is countable.
- The random variable in this case is ***discrete***.
- We define its probability law by its probability distribution.
- We will define the two main numerical characteristics of a discrete random variable:
  - Expected value: characteristic of centrality (the *mean*).
  - Variance: characteristic of dispersion.
- We will also define the couples of random variables.

Discrete Random Variables

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- $B \subset \mathbb{R}$, $P(X \in B) = \sum_{i/x_i \in B} p(x_i)$.

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- $p(x_i) \geqslant 0$ and $\sum_{i=1}^{\infty} p(x_i) = 1$.
- If the number of possible values of X is small enough, the probability distribution of X is often presented as a table.

$P(X \in [0,6]) = P(X=1) + P(X=5) = 1/3 + 1/2$ in the prev. example

Definition

Given a discrete random variable X, we call cumulative distribution function of X (or simply distribution function), denoted F_X, the function defined by: for any real a,

$F_X : \mathbb{R} \to \mathbb{R}$ ([0,1])

$a \in \mathbb{R}$,

$$F(a) = P(X \leqslant a) = \sum_{i / x_i \leqslant a} P(X = x_i)$$

The value $F_X(a)$ represents the probability that X takes a value smaller or equal to a.

$$F_X(a) = P(X \leq a)$$

$P : \mathcal{A} \to [0,1]$

$X : \Omega \to \mathbb{R}$

$F_X : \mathbb{R} \to [0,1]$

$F_X(0) = P(X \leq 0) = P(X = -10)$

$F_X(2) = P(X \leq 2) = P(X = -10) + P(X = 1) \ldots$

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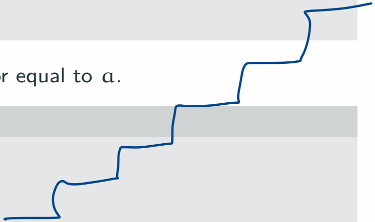

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The distribution function characterizes the distribution of X. In other words, if X and Y are two random variables, we have $F_X = F_Y$ if and only if their probability distributions are the same.

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Remark

One can compute the individual probabilities by:

$$p_i = P(X = x_i) = F(x_i) - F(x_{i-1}) \qquad \text{pour } 1 \leqslant i \leqslant n$$

Example

We play three times to “heads or tails”

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- $\Omega = \{\mathrm{P}, \mathrm{F}\}^3$.

P, F, F
PPP
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⋮

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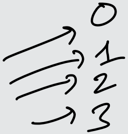

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T H H
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Distribution function and probabilities over X

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Using the same method we obtain the probability distribution of X:

k	0	1	2	3
$P(X = k)$	$1/8$	$3/8$	$3/8$	$1/8$

Distribution function and probabilities over X

The distribution function X is therefore given by:

$$F(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1/8 & \text{si } 0 \leqslant x < 1 \\ 1/2 & \text{si } 1 \leqslant x < 2 \\ 7/8 & \text{si } 2 \leqslant x < 3 \\ 1 & \text{si } x \geqslant 3 \end{cases}$$

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One can represent both the probability distribution and the distribution function of X in the same table:

k	0	1	2	3
$P(X = k)$	1/8	3/8	3/8	1/8
$F_X(x)$	1/8	1/2	7/8	1

The graph of the distribution function is represented below:

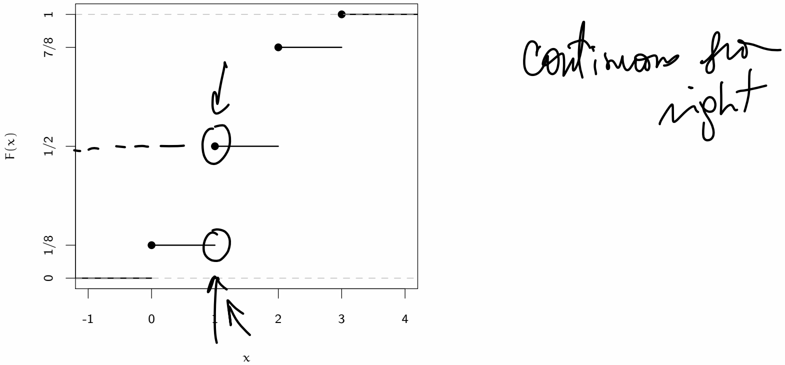

Figure 1: Distribution function

Here is another slightly different representation of the distribution function:

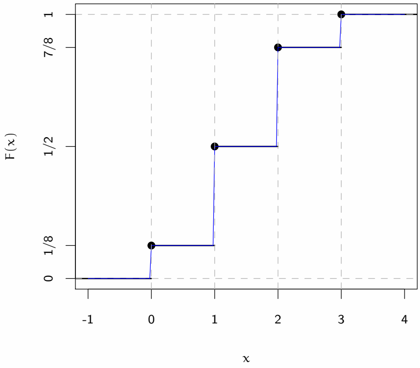

Figure 2: Distribution function

The indicator random variable

Definition

Let A an event. We call indicator random variable of the event A, the random variable $X = \mathbb{1}_A$ defined by:

$$X(\omega) = \begin{cases} 1 & \textit{si } \omega \in A \\ 0 & \textit{si } \omega \in \bar{A} \end{cases}$$

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The distribution function of the indicator random variable is therefore:

$$F(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1 - p & \text{si } 0 \leqslant x < 1 \\ 1 & \text{si } x \geqslant 1 \end{cases}$$

Example

- Let $\mathbb{U}$ an urn containing two white ball and three red balls.
- We randomly take one ball out of the box.
- Let A : “take one white ball out”.
- Let X be the indicator random variable of A.

Find the probability distribution and the distribution function of X.

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The probability distribution of X is

k	0	1
$P(X=k)$	$\frac{3}{5}$	$\frac{2}{5}$

and its distribution function is:

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Moments of a discrete random variable

Definition

For a discrete random variable X *with probability distribution* $p(.)$, *we define the expected value of* X, *called* $E(X)$, *by*

$$E(X) = \sum_{i \in \mathbb{N}} x_i p(x_i)$$

Think about Means

Expected value

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In concrete terms, the expected value of X *is the weighted mean of the values of* X*, the weights being the probabilities associated to the values of* X*.*

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Examples

1. In the previous example where we play three times to "heads or tails", the expected value of X is:

$$E(X) = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = 1.5$$

2. For the indicator random variable of A:

$$E(X) = 0 \times P(X = 0) + 1 \times P(X = 1) = P(A) = p$$

which means that the expected value of the indicator of an event A corresponds to the probability that A occurs.

Expected value of a function of a random variable

Theorem

Let X be a discrete random variable whose possible values are x_i, $i \geqslant 1$, and denote by $p(x_i)$ the probability that $X = x_i$ occurs. Then, for any real function g, we have

Transfer theorem

$$E(g(X)) = \sum_i g(x_i)p(x_i)$$

$$E(X^2) = \sum x_i^2 \, P(X = x_i)$$

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Example

Let X be a random variable that can take three values $\{-1, 0, 1\}$ with the following probabilities:

$$P(X = -1) = 0.2 \qquad P(X = 0) = 0.5 \qquad P(X = 1) = 0.3$$

Calculate $E(X^2)$.

Solution

Solution

First method: Let $Y = X^2$.

Second method: Using the theorem

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First method: Let $Y = X^2$. The probability distribution of Y is given by

$$P(Y = 1) = P(X = -1) + P(X = 1) = 0.5$$
$$P(Y = 0) = P(X = 0) = 0.5$$

So

$$E(X^2) = E(Y) = 1(0.5) + 0(0.5) = 0.5$$

Second method: Using the theorem

Solution

First method: Let $Y = X^2$. The probability distribution of Y is given by

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$$E(X^2) = E(Y) = 1(0.5) + 0(0.5) = 0.5$$

Second method: Using the theorem

$$E(X^2) = (-1)^2(0.2) + 0^2(0.5) + 1^2(0.3)$$
$$= 1(0.2 + 0.3) + 0(0.5) = 0.5$$

Solution

First method: Let $Y = X^2$. The probability distribution of Y is given by

$$P(Y = 1) = P(X = -1) + P(X = 1) = 0.5$$
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Remark

$$0.5 = E(X^2) \neq (E(X))^2 = 0.01$$

Linearity of the expected value

Properties

1. $E(X+a) = E(X) + a, \quad a \in \mathbb{R}$

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All these three properties are summarised in the claim that the expected value is linear:

$$E(\lambda X + \mu Y) = \lambda E(X) + \mu E(Y), \quad \forall \lambda \in \mathbb{R}, \forall \mu \in \mathbb{R}.$$

Definition

Let X be a discrete random variable. We call variance of X, denoted $V(X)$, the quantity defined by, when it exists,

$$V(X) = E\left[(X - E(X))^2\right]$$

Thus, the variance is the expected value of the square of the centered random variable $X - E(X)$. The variance can be interpreted as a measure of the dispersion of the possible values of X around its expected value.

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Indeed:

$$\begin{aligned} V(X) &= E\left[X^2 - 2XE(X) + E^2(X)\right] \\ &= E(X^2) - E[2XE(X)] + E[E^2(X)] \\ &= E(X^2) - 2E^2(X) + E^2(X) \end{aligned}$$

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Previously, we saw that $E(X) = \frac{7}{2}$. Moreover,

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And therefore

$$\begin{aligned} V(X) &= E(X^2) - E^2(X) \\ &= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12} \end{aligned}$$

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Definition

*Let X be a discrete random variable. The square root of the variance is called the **standard deviation** of X and is denoted*

$$\sigma_X = \sqrt{V(X)}$$

σ_X *has the same physical units as the random variable* X.

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- The smaller sigma is, the closer to each other the values of the data are.
- Example: the dispersion of the grades in an exam. The smaller sigma is, the more homogeneous the class is.
- - Expected value and standard deviation are linked through *Bienaymé-Tchebychev inequality*.

Here

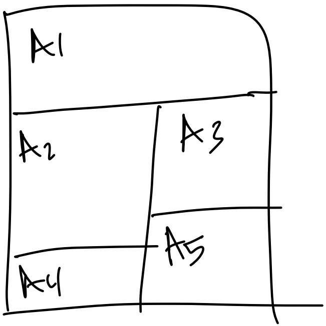
Bienaymé-Tchebychev inequality

Theorem

Let X *a random variable of expected value* μ *and variance* σ^2. *For all* $\varepsilon > 0$, *We have:*

$$P\left(|X - E(X)| \geqslant \varepsilon\right) \leqslant \frac{\sigma^2}{\varepsilon^2}$$

$$P\left(\bigcup_i A_i\right) = \sum P(A_i)$$



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This inequality can be written in a slightly different fashion. Let $k = \varepsilon/\sigma$.

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Importance

This inequality relates the probability for X to deviate from its expected value $E(X)$ to its variance, which is precisely an indicator of the dispersion around the expected value. The inequality makes quantitatively precise the statement "the smaller the variance is, the less likely it is to find values far away from the expected value".

Definition

We call non centered moment of order $r \in \mathbb{N}^*$ *of* X *the quantity, when it exists:*

$$m_r(X) = \sum_{i \in \mathbb{N}} x_i^r p(x_i) = E(X^r).$$

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The centered moment of order $r \in \mathbb{N}^*$ *the quantity, when it exists:*

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Remark

The first moments are:

- $m_1(X) = E(X), \quad \mu_1(X) = 0.$
- $\mu_2(X) = V(X) = m_2(X) - m_1^2(X).$

Two Random Variables

So far, we have dealt with one random variable. However, it is often necessary to consider events related to two variables simultaneously, or even to more than two variables.

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Definition

Let X *and* Y *two discrete random variables, defined on probability space* $(\Omega, \mathcal{A}, P)$ *and that* $X(\Omega) = \{x_1, x_2, \dots, x_l\}$ *and* $Y(\Omega) = \{y_1, y_2, \dots, y_k\}$, l *and* $k \in \mathbb{N}$.

The ***probability law of*** (X, Y) *is defined by joint probabilities:*

$$p_{ij} = P(X = x_i; Y = y_j) = P(\{X = x_i\} \cap \{Y = y_j\})$$

We have

$$p_{ij} \geqslant 0 \quad \textit{et} \quad \sum_{i=1}^{l} \sum_{j=1}^{k} p_{ij} = 1$$

The pair (X, Y) *is called two dimensional random vector and can have* $l \times k$ *valeurs.*

The probabilities p_{ij} can be presented in a two dimensional table than we call joint probability distribution table:

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Table 1: Joint probability distribution table

$X\backslash Y$	y_1	y_2	$\dots$	y_j	$\dots$	y_k
x_1	p_{11}	p_{12}		p_{1j}		p_{1k}
x_2	p_{21}	p_{22}		p_{2j}		p_{2k}
$\vdots$						
x_i	p_{i1}	p_{i2}		p_{ij}		p_{ik}
$\vdots$						
x_l	p_{l1}	p_{l2}		p_{lj}		p_{lk}

In the header we have the possible values of Y and in the first column the possible values of X. The probability $p_{ij} = P(X = x_i; Y = y_j)$ is at the intersection of i^{th} line and j^{th} column.

Example

Three balls are drawn at random from an urn containing 3 red, 4 white and 5 black balls. X and Y are respectively the number of red and white balls drawn. Determine the joint probability distribution of the pair (X, Y).

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- $|\Omega| = C_{12}^{3} = 220$.
- $X(\Omega) = \{0, 1, 2, 3\}$ and $Y(\Omega) = \{0, 1, 2, 3\}$.

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- $p(1, 0) = C_3^1 C_5^2 / C_{12}^3 = \frac{30}{220}$.

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Table 2: Joint probability distribution table

$X\backslash Y$	0	1	2	3
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0
3	$\frac{1}{220}$	0	0	0

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- Marginal distribution of X:

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- Marginal distribution of Y:

$$p_{.j} = P(Y = y_j) = P[\Omega \cap \{Y = y_j\}] = \sum_{i=1}^{l} p_{ij} \qquad \forall\, j = 1, 2, \dots, k$$

We can calculate the marginal distributions directly from the table of the joint distribution.

Table 3: Joint distribution table with marginal distributions

$X\backslash Y$	y_1	y_2	$\dots$	y_j	$\dots$	y_k	Marginal of X
x_1	p_{11}	p_{12}		p_{1j}		p_{1k}	$p_{1.}$
x_2	p_{21}	p_{22}		p_{2j}		p_{2k}	$p_{2.}$
$\vdots$							
x_i	p_{i1}	p_{i2}		p_{ij}		p_{ik}	$p_{i.}$
$\vdots$							
x_l	p_{l1}	p_{l2}		p_{lj}		p_{lk}	$p_{l.}$
Marginal of Y	$p_{.1}$	$p_{.2}$		$p_{.l}$		$p_{.k}$	1

Example: Marginal Probability Distributions

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Table 4: Joint distribution table

$X \backslash Y$	0	1	2	3	$p_{i.} = P(X = x_i)$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
$p_{.j} = P(Y = y_j)$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	1

Definition

For each value y_j *of* Y *such that* $p_{.j} = P(Y = y_j) \neq 0$ *we can define the conditional distribution of* X *given* $Y = y_j$ *by*

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Same for Y *given* $X = x_i$*:*

$$p_{j/i} = P(Y = y_j / X = x_i) = \frac{P(X = x_i ; Y = y_j)}{P(X = x_i)} = \frac{p_{ij}}{p_{i.}} \qquad \forall j = 1, 2, \ldots, k$$

Definition

We say that two random variables are independent iff

$$P(X = x_i; Y = y_j) = P(X = x_i)P(Y = y_j) \qquad \forall i = 1, 2, \dots, l \text{ and } j = 1, 2, \dots, k$$

One demonstrates that

$$P(\{X \in A\} \cap \{Y \in B\}) = P(\{X \in A\})P(\{Y \in B\}) \qquad \forall\ A \text{ and } B \in \mathcal{A}$$

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Properties

Let two random variables X and Y,

1. $E(X + Y) = E(X) + E(Y)$
2. If X and Y are independent so $E(XY) = E(X)E(Y)$. But the reciprocal is not always true.

Definition

Let two random variables X *and* Y*. The* ***covariance*** *of* X *and* Y*, when it exists, is*

$$Cov(X,Y) = E[(X - E(X))(Y - E(Y))] = \sum_i \sum_j (x_i - E(X))(y_j - E(Y))p_{ij}$$

that we can calculate using the formula

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

Definition

*Let two random variables X and Y. The **covariance** of X and Y, when it exists, is*

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))] = \sum_i \sum_j (x_i - E(X))(y_j - E(Y))p_{ij}$$

that we can calculate using the formula

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- If X and Y are independant so
  - $Cov(X,Y) = 0$ (the reciprocal is not always true)
  - $V(X+Y) = V(X) + V(Y)$ (the reciprocal is not always true)

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The correlation between X and Y is defined by

$$\rho = \rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{V(X)V(Y)}} = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

We can demonstrate that

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Interpretation of ρ

- The correlation coefficient is a measure of the degree of linearity between X and Y.
- Values of rho close to 1 or -1 indicate an almost rigorous linearity between X and Y.
- Values of rho close to 0 indicate the absence of any linear relationship.
- When $\rho(X,Y)$ is positive, Y tends to increase if X does the same.
- When $\rho(X,Y) < 0$, Y tends to decrease if X increases.

Linear correlation